Section 3.3: Applications of Linear Functions

Since we have already studied lines and linear equations extensively, we can now focus on how to use linear functions to solve problems. We will focus on two main applications. First, we look at the expressions for revenue, cost, and profit that we studied in Chapter 1, and view them now as functions. Secondly, we will take a look at supply and demand, which is a major concept in economics.

Revenue, Cost, and Profit Functions.
Recall from Chapter 1 that:

- Revenue = (Price per item) × (Number of items)
- Cost = Fixed Costs + Variable Costs (which depend on the number of items)
- Profit = Revenue − Cost

Therefore, all three of these depend on the number of items produced and sold. Therefore, we can let the variable x represent the number of items produced and sold, and think of these three quantities as functions of x.

Example 1. Suppose a company sells wallets for \$30 each. Their fixed costs are \$2700, and the cost of manufacturing each wallet is \$12. Find formulas for the revenue, cost, and profit as functions of the number of wallets produced and sold.

Answer. We let x be the number of wallets produced and sold. Let $R(x)$ be the revenue made by selling x wallets. Let $C(x)$ be the total cost of producing x wallets. Let $P(x)$ be the profit made from producing and selling x wallets.

Then since the price per wallet is \$30, we get that $R(x) = 30x$.
Since the cost of manufacturing each wallet is \$12, the variable cost of producing x wallets is $12x$. When we add the fixed costs, we get that $C(x) = 12x + 2700$.
We subtract these quantities to get the profit. That is, $P(x) = R(x) - C(x)$, so

$$\begin{aligned} P(x) &= (30x) - (12x + 2700) \\ &= 30x - 12x - 2700 \\ &= 18x - 2700 \end{aligned}$$

That is, $P(x) = 18x - 2700$.

We can use these functions to decide how many items must be produced and sold in order to start making a profit. This is called "Break-Even Analysis." The number of items that a company must sell to make their revenue equal to the total cost is called the *break-even point*. If the company sells more items than this, they will make a profit.

Example 2. Find the break-even point for the wallet company from the previous example.

Answer. We found that the profit function is $P(x) = 18x - 2700$. The break-even point is the value of x that will make the profit function equal to 0. Therefore, we solve $18x - 2700 = 0$.
$18x - 2700 = 0 \implies 18x = 2700 \implies x = 150$
Therefore, the break-even point is 150 wallets. The company must sell more than 150 wallets to make a profit.

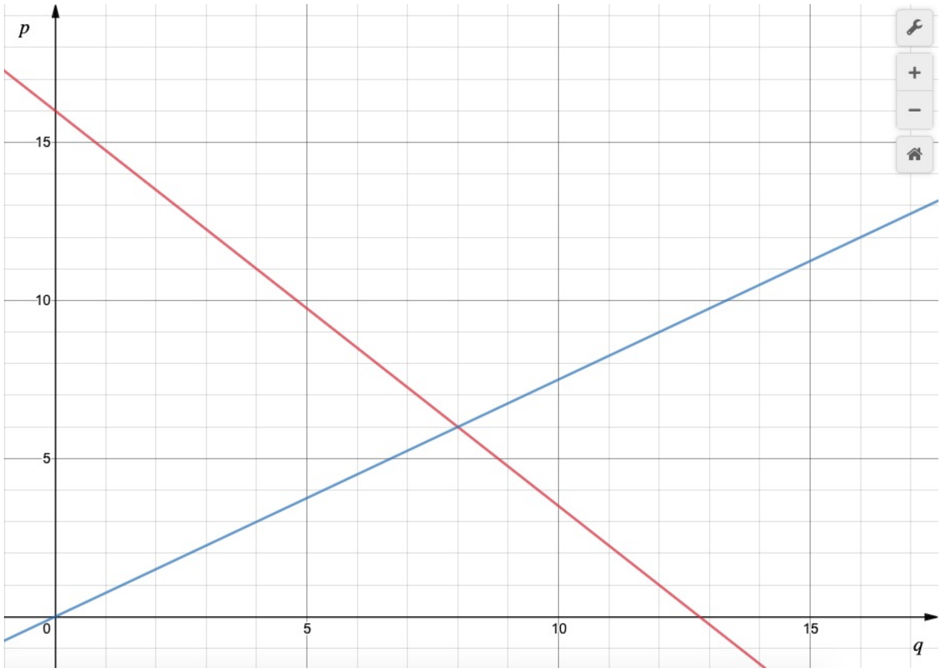
Supply and Demand.
In all our revenue/cost/profit problems, we've assumed that every item produced ends up being sold. In real life, this isn't the case all the time. If a product is priced too high, very few people will want to buy it. If a product is priced very low, there may be many people who want to buy it, but the manufacturer may not be willing to produce it for such low revenue per item. This is the idea behind *supply and demand.*

That is, the quantity of items that the consumers demand and the quantity of items that the manufacturer are willing to supply both relate to the price of the item. Let's introduce some variables. Let q be the quantity of items, and let p be the price of the item.

A *supply curve* gives a relationship between p and q that expresses the sale price p at which the supplier is willing to produce the quantity q.
A *demand curve* gives a relationship between p and q that expresses the sale price at which consumers would be willing to purchase the quantity q.
We can graph these relationships on a plane, with q on the x-axis and p on the y-axis.

Example 3. Suppose the demand curve for a brand of shampoo is $p = 16 - \frac{5}{4}q$, and suppose the supply curve for the shampoo is given by $p = \frac{3}{4}q$. (Note: In practice, the units for quantity will often be "thousands of items" or "millions of items".) To use these curves, we can plug in different values for q or p. For example, this says that in order to find the price at which consumers would be willing to buy only 4 units of the product, we plug $q = 4$ into the demand curve, and we find that $p = 16 - \frac{5}{4}(4) = 11$. So at the price of \$11, consumers would only be willing to buy 4 units. Similarly, by plugging $q = 4$ into the supply curve, we can find the price at which suppliers would be willing to produce 4 units. Using the supply curve, we get $q = \frac{3}{4}(4) = 3$. So at the price of \$3, suppliers would only be willing to produce 4 units. In general, as the number of units increases, it will require a higher price to convince the supplier to make that many, and it will require a lower price to convince consumers to purchase that many.

We can graph these curves on a qp-plane.



The red line gives the demand curve, and the blue line gives the supply curve.

Note that these lines intersect. This gives us a point at which supply and demand are equal. We call this the *equilibrium point*. At this point, we call the value of q the *equilibrium quantity* and we call the value of p the *equilibrium price*. We can find the equilibrium point by setting the supply curve equal to the demand curve and solving for q. This will give us the equilibrium quantity.

In the previous example, since the supply curve is $p = \frac{3}{4}q$ and the demand curve is $p = 16 - \frac{5}{4}q$, we solve the equation $\frac{3}{4}q = 16 - \frac{5}{4}q$.

$\frac{3}{4}q = 16 - \frac{5}{4}q \Longrightarrow \frac{3}{4}q + \frac{5}{4}q = 16 \Longrightarrow 2q = 16 \Longrightarrow q = 8$

Therefore, the equilibrium quantity is 8 units. To find the equilibrium price, we can plug $q = 8$ into either equation for supply or demand. For example, using the supply curve, we find $p = \frac{3}{4}(8)$, so $p = 6$. That means the equilibrium price is \$6. If we look at the graph, we see this verifies that the intersection point is $(8, 6)$.

Final note: In practice, both the topics in this section can be generalized beyond linear functions. That is, cost functions, supply curves, and demand curves may not necessarily be linear. In those cases, all of the same definitions from this section still hold, but we may need to solve more complicated equations in order to find the information we want.

SUMMARY:

- The break-even point is the number of items that must be produced and sold in order to make the profit 0.
- Supply and demand curves relate the quantity of items to the price of the item.
- The equilibrium point is the point at which supply and demand are equal, and its coordinates are the equilibrium quantity and the equilibrium price.