

### SECTION 4.3: LOGARITHMS

As noted at the end of the Section 4.2 notes, we need a way to undo exponentiation. We use subtraction to undo addition. We use division to undo multiplication. We can use the  $n$ th root to undo raising something to the  $n$ th power. But if the variable is in the exponent, we can't use radicals to undo them. We have to define something new to undo exponentiation in this way.

**Definition.** The *logarithm of base  $a$* , denoted  $\log_a$ , is a function that undoes the exponential function of base  $a$ . That is,

$$\log_a(x) = y \text{ means } a^y = x.$$

So to start, we can think of logarithms as a way to translate an exponential equation into a different form.

#### Example 1.

- We know that  $2^3 = 8$ . This means that  $\log_2(8) = 3$ .
- We know that  $10^2 = 100$ . This means that  $\log_{10}(100) = 2$ .
- We know that  $3^{-2} = \frac{1}{9}$ . This means that  $\log_3\left(\frac{1}{9}\right) = -2$ .
- We know that  $16^{1/2} = 4$ . This means that  $\log_{16}(4) = \frac{1}{2}$ .
- Since  $a^0 = 1$  no matter what  $a$  is, we also have that  $\log_a(1) = 0$  no matter what  $a$  is.

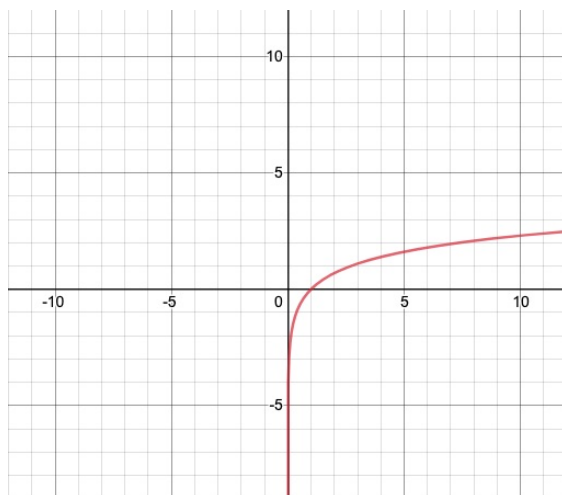
From this last item, we see that we can use some of the properties of exponential functions to get properties of logarithms. In particular, we get the following properties:

- (a)  $\log_a(x)$  is defined for any  $a > 0$  with  $a \neq 1$ , and any  $x > 0$ . We can not make sense of  $\log_a$  if  $a \leq 0$ . We say that  $\log_a(x)$  is undefined if  $x \leq 0$ .
- (b) For any base  $a$ ,  $\log_a(1) = 0$ .
- (c) For any base  $a$ ,  $\log_a(a) = 1$ .
- (d) For any base  $a$  and any real number  $r$ , we have that  $\log_a(a^r) = r$ .
- (e) For any base  $a$  and any  $x > 0$ , we have that  $a^{\log_a(x)} = x$ .

These last two properties really illustrate how logarithms and exponential functions undo each other. That is, if  $f(x) = a^x$  and  $g(x) = \log_a(x)$ , we have that  $f(g(x)) = x$  and that  $g(f(x)) = x$ . This is what it means for two functions to be inverses of each other. We'll use this to solve exponential and logarithmic equations in Section 4.4.

For now, let's get some more properties of logarithmic functions as functions themselves. First, since  $\log_a(x)$  is undefined when  $x \leq 0$ , this tells us that the domain of  $\log_a$  is the interval  $(0, \infty)$ . Note that this gives us another domain restriction to worry about! We should place “we can't take a log of a negative number or 0” on the list of domain restrictions beside “we can't divide by 0” and “we can't take the square root of a negative number”.

Since logarithms and exponential functions undo each other, it makes sense that their graphs would be related. In particular, if the point  $(x, y)$  is on the graph of  $f(x) = a^x$ , then the point  $(y, x)$  is on the graph of  $g(x) = \log_a(x)$ . This means that the graph of  $\log_a$  is a sort of “mirror image” of the graph of  $a^x$ , reflected through the diagonal line  $y = x$ . Under this reflection, horizontal lines get reflected to vertical lines, so the horizontal asymptote that  $a^x$  has at  $y = 0$  becomes a *vertical asymptote* at  $x = 0$ . Below is the graph of  $f(x) = \log_e(x)$ .



Recall that as the base  $a$  increases, that meant that the graph of  $a^x$  becomes steeper. Translating that property to the reflection, it means that the graph of  $\log_a(x)$  becomes less steep as the base  $a$  increases.

The same important bases that we mentioned in Section 4.1 are still important here: 10, 2, and  $e$ . The importance of base 2 is still largely confined to computer science or other technical uses. Using the base 10 used to be extremely important, since before the creation of mechanical calculators or computers, many difficult calculations were done by converting to  $\log_{10}$ . In particular, this is how slide rules were used to do math a century ago. It was so widely used that it became shorthand to just say log to specifically mean  $\log_{10}$ . Your calculator today still may have a button that just says “log”, and that button computes  $\log_{10}$ . Although  $\log_{10}$  has become a less crucial part of computational mathematics now that we use calculators instead of slide rules, it is still used in some areas of math for things like presenting data.

But by far, the most useful base of a logarithm is  $e$ , and we will use a special notation for it.

**Definition.** The logarithm with base  $e$  is called the *natural logarithm*, denoted by  $\ln$ . That is,  $\ln(x) = \log_e(x)$ .

Your calculator likely has a button that says “ln”. You can use it to make computations involving the natural logarithm.

**Example 2.** Using a calculator, we find that  $\ln(462)$  is approximately 6.135564891. We’ll usually round to an appropriate number of decimals. This means that  $e^{6.1356} \approx 462$ . You can also check this on your calculator.

We’ve already translated a couple properties of exponentiation into properties of logarithms. Let’s recall three more important properties of exponentiation and see what they tell us about logarithms.

### Rules of Exponentiation

- If we multiply two exponential expressions with the same base, then the exponents add. That is,  $a^x a^y = a^{x+y}$ .
- If we divide two exponential expressions with the same base, then the exponents subtract. That is,  $\frac{a^x}{a^y} = a^{x-y}$ .
- If we raise an exponential expression to another power, the exponents multiply. That is,  $(a^x)^y = a^{xy}$ .

We then translate these to get product, quotient, and power properties for logarithms. I call these our “log laws”.

### Log Laws

- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$
- $\log_a(x^r) = r \log_a(x)$

To restate these in words, we have that the log of a product is the sum of the logs (so logarithms turn multiplication into addition). We have that the log of a quotient is the difference of the logs (so logarithms turn division into subtraction). And we have that the log of a power is the product of the power with the log (so logs turn exponentiation into multiplication).

**Example 3.** Compute  $\log_2(48) - \log_2(3)$ .

**Answer.** Note that at this point, we can’t yet compute either  $\log_2(48)$  or  $\log_2(3)$  on their own, since neither 48 nor 3 is a power of 2 that we know. However, using the second log law, we see that

$$\log_2(48) - \log_2(3) = \log_2\left(\frac{48}{3}\right) = \log_2(16) = 4.$$

Thus,  $\log_2(48) - \log_2(3) = 4$ .

We can do the same process we did in the previous example using variables instead of numbers.

**Example 4.** Write the following as a single logarithm expression:

$$2 \log_3(x) + \log_3(x + 1) - 3 \log_3(y).$$

**Answer.** To start, we use the third log law to put the constant multiples inside the logs as exponents. That is, we get

$$\log_3(x^2) + \log_3(x + 1) - \log_3(y^3).$$

Then we can use the first two log laws to combine these, and we get

$$\log_3(x^2) + \log_3(x + 1) - \log_3(y^3) = \log_3\left(\frac{x^2(x + 1)}{y^3}\right).$$

So,  $\log_3\left(\frac{x^2(x + 1)}{y^3}\right)$  is a single logarithmic expression that is equal to the original sum and difference of expressions.

We can also use this in reverse, to expand out a single logarithmic expression as a sum or difference of simpler logarithmic expressions.

**Example 5.** Use the log laws to expand out the expression  $\ln\left(\frac{x^2y}{4\sqrt{z}}\right)$  in a form with no logarithm of a product, quotient, or power.

**Answer.** The most important thing to be careful about here is to pay attention to which variables are getting hit by which exponents. A common mistake is to pull an exponent out incorrectly due to a misuse of the third log law. For that reason, I always recommend that you save the third log law for last when you are expanding an expression like this. So let's start with the second log law, and get rid of the division by rewriting it as subtraction of the logs.

$$\ln\left(\frac{x^2y}{4\sqrt{z}}\right) = \ln(x^2y) - \ln(4\sqrt{z}).$$

Next, we can replace every product inside a log with a sum of logs. That is, we can write

$$\ln(x^2y) = \ln(x^2) + \ln(y) \text{ and } \ln(4\sqrt{z}) = \ln(4) + \ln(\sqrt{z}).$$

We might also notice that  $4 = 2 \times 2$ , so we can write  $\ln(4) = \ln(2) + \ln(2)$ .

Now, none of the individual logs remaining has any multiplication or division happening inside the parentheses, except for exponentiation. So we can use the third log law now when it applies. We have that

$$\ln(x^2) = 2 \ln(x) \text{ and that } \ln(\sqrt{z}) = \ln(z^{1/2}) = \frac{1}{2} \ln(z).$$

Finally, let's put it all together.

$$\begin{aligned}
 \ln\left(\frac{x^2y}{4\sqrt{z}}\right) &= \ln(x^2y) - \ln(4\sqrt{z}) \\
 &= (\ln(x^2) + \ln(y)) - (\ln(4) + \ln(\sqrt{z})) \\
 &= (\ln(x^2) + \ln(y)) - (\ln(2) + \ln(2) + \ln(\sqrt{z})) \\
 &= (\ln(x^2) + \ln(y)) - (2\ln(2) + \ln(\sqrt{z})) \\
 &= \ln(x^2) + \ln(y) - 2\ln(2) - \ln(\sqrt{z}) \\
 &= 2\ln(x) + \ln(y) - 2\ln(2) - \frac{1}{2}\ln(z)
 \end{aligned}$$

This is the answer we want.

As we can see, the log laws let go back and forth between having a single log with a complex expression inside and having a complex combination of many simple logs. Both are useful in different situations, so it's important to be able to freely go between both.

In Example 3, we noted that we don't yet know how to compute  $\log_2(48)$  or  $\log_2(3)$ . While your calculator has a button for  $\ln$  and a button for  $\log_{10}$  (which is just denoted as LOG on the calculator), most calculators don't have a button for  $\log_2$ , or  $\log_5$ , or  $\log_{4.318}$ . We want to be able to get decimal estimates for these numbers somehow.

The answer is to use the calculator buttons we DO have, and a bit of trickery using the log laws. Let's say we want to compute  $\log_2(3)$ . I'll use the letter  $y$  to denote this unknown number for now. That is,  $y = \log_2(3)$ .

But then we can rewrite this statement as an exponent statement. That is,  $2^y = 3$ .

Now, we apply a log to both sides that we can compute with our calculator. So I'll use  $\ln$ , but we could also use  $\log_{10}$  here instead. This gives us that  $\ln(2^y) = \ln(3)$ .

But now we can use our third log law on the left to pull the exponent out as a multiple. In other words,  $\ln(2^y) = y\ln(2)$ . Therefore,  $y\ln(2) = \ln(3)$ .

But now we can solve this for  $y$ , and we find that  $y = \frac{\ln(3)}{\ln(2)}$ . But we can get decimal expansions for both numerator and denominator using our calculator! In particular,

$$y = \log_2(3) = \frac{\ln(3)}{\ln(2)} = \frac{1.098612289}{0.6931471806} = 1.5849625.$$

This is how we get the change of base formula.

### Change of Base Formula

For any base  $a$ , we have that  $\log_a(x) = \frac{\ln(x)}{\ln(a)}$ .

Now, we can use our calculator to compute any logarithm that we want with a high level of precision.

**Example 6.** Compute  $\log_{16}(10)$ .

**Answer.** Using the change of base formula,

$$\log_{16}(10) = \frac{\ln(10)}{\ln(16)} = \frac{2.302585093}{2.772588722} = 0.830482.$$

SUMMARY:

- A logarithm of base  $a$  “undoes” an exponential expression of base  $a$ . That is,  $\log_a(x) = y$  and  $a^y = x$  mean the same thing.
- The function  $f(x) = \log_a(x)$  has a domain of  $(0, \infty)$ . That is, we can’t take a logarithm of a negative number or zero, no matter what the base is.
- Laws of Logarithms help us rewrite logarithm expressions in either an expanded out form or a single log form, depending on what we need.
- We denote the logarithm with base  $e$  with the notation  $\ln$ , and can use the calculator button for  $\ln$  in our computations.
- We can use the change of base formula to compute any logarithm using our calculator.