Section 4.4: Exponential and Logarithmic Equations

Now that we know how to undo exponentials, we can use this tool to solve exponential equations.

To start, consider the equation $4^x = 16$. We might not even need logarithms to solve this, since we only need to think what value of the exponent x will make this equation true, and we realize that $x = 2$.

Let's make this a little tougher by next considering the equation $4^x = 10$. Now since 10 is not a perfect power of 4, we can't use the same method. However, we can use logarithms to help us now, because we know that $\log_4$ undoes the exponential function with base 4. That is, $\log_4(4^x) = x$. So starting with $4^x = 10$, if we apply $\log_4$ to both sides, we find that $\log_4(4^x) = \log_4(10)$. Therefore, $x = \log_4(10)$. Note that in this case, getting the answer is as simple as writing the translation of the original equation in terms of logarithms!

However, we want to be able to estimate this answer as a decimal expression, so we can use our change of base formula to say that

$$x = \log_4(10) = \frac{\ln(10)}{\ln(4)} = 1.660964.$$

Hopefully this method makes sense, because it models the way we solve lots of other equations. We undo addition with subtraction, we undo multiplication with division, and now we undo exponentiation with logarithms.

However, the downside is that we'll often have to perform this last step of using the change of base formula. The better way to do this is to just always apply the natural logarithm, ln. This doesn't perfectly undo the exponential in the same way, but by the third log law, it lets us take out the exponent as a multiple! Let's see how this works:

Starting from $4^x = 10$, we apply ln to both sides. This gives us

$$\ln(4^x) = \ln(10).$$

Now we can use the third log law to manipulate the left side, which tells us that $\ln(4^x) = x\ln(4)$. Therefore,

$$x\ln(4) = \ln(10).$$

But now, this is a linear equation!!! Sure, $\ln(4)$ and $\ln(10)$ are ugly numbers, but they're still just numbers. To solve for x, we just divide both sides by $\ln(4)$, and we get $x = \dfrac{\ln(10)}{\ln(4)}$, just as before.

This last method will be our general tool for solving exponential equations, which are equations in which the variable appears in the exponent of an expression.

General Strategy for Solving Exponential Equations

(1) Isolate the expression with the variable in the exponent.

(2) Apply ln to both sides.

(3) Use log laws to simplify.

(4) Solve the resulting equation for the variable.

Example 1. Solve the equation $3^{x+5} - 5 = 12$.

Answer. We apply our strategy. First, we want to isolate the 3^{x+5} piece. In particular, we want to get rid of that -5 on the left, so we add 5 to both sides, which gives us $3^{x+5} = 17$.

Then we can apply ln to both sides. Therefore, we get $\ln(3^{x+5}) = \ln(17)$. Using the third log law, we then get $(x+5)\ln(3) = \ln(17)$. Again, this is just a linear equation! We can either expand out first, or divide both side by $\ln(3)$ first. I'll do the latter here. Dividing both sides by $\ln(3)$, we get $x + 5 = \dfrac{\ln(17)}{\ln(3)}$. Then we can solve for x by subtracting 5 from both sides, and we find that

$$x = \frac{\ln(17)}{\ln(3)} - 5 = -2.421.$$

Example 2. Solve the equation $e^{5x} = 12$.

Answer. The exponential expression is already isolated, so we can take ln of both sides.

$$\begin{aligned} & e^{5x} = 12 \\ \implies\quad & \ln(e^{5x}) = \ln(12) \\ \implies\quad & 5x\ln(e) = \ln(12) \end{aligned}$$

But now we might remember that $\ln(e) = 1$, since ln is the logarithm with base e! Therefore, all we have to do is divide by 5, and we get $x = \dfrac{\ln(12)}{5} = .49698$.

Example 3. Solve the equation $4(6^{2x-1}) = 28$.

Answer. To isolate the exponential expression, we can start by dividing both sides by 4. This gives us $6^{2x-1} = 7$. We apply ln to both sides and use the third log law to get

$$\begin{aligned} & 6^{2x-1} = 7 \\ \implies\quad & \ln(6^{2x-1}) = \ln(7) \\ \implies\quad & (2x-1)\ln(6) = \ln(7) \end{aligned}$$

As in Example 1, we can either expand out first or divide both sides by $\ln(6)$. This time, I'll expand out by distributing.

$$2x\ln(6) - \ln(6) = \ln(7)$$

We add $\ln(6)$ to both sides and then divide both sides by $2\ln(6)$.

$$\begin{aligned} 2x\ln(6) &= \ln(7)+\ln(6) \\ \implies \qquad x &= \frac{\ln(7)+\ln(6)}{2\ln(6)} = 1.043 \end{aligned}$$

The last type of exponential equation we could encounter is one in which there is one exponential expression on each side. Again, our strategy here is to apply ln to both sides and use the log laws to simplify.

Example 4. Solve the equation $2^{3x} = 5^{3-2x}$.

Answer. First, we take the natural log of both sides:

$$\ln(2^{3x}) = \ln(5^{3-2x})$$

Then we use the third log law to pull out the exponents:

$$3x\ln(2) = (3-2x)\ln(5)$$

We can use the distributive property to simplify:

$$3x\ln(2) = 3\ln(5) - 2x\ln(5)$$

Now this is a linear equation! Let's put all terms with an x on the same side. We do this by adding $2x\ln(5)$ to both sides.

$$3x\ln(2) + 2x\ln(5) = 3\ln(5)$$

Now since both terms on the left have an x, we can factor out the x. This gives us:

$$x(3\ln(2) + 2\ln(5)) = 3\ln(5)$$

Then the last step to isolate x is to divide by $(3\ln(2) + 2\ln(5))$, and we get

$$x = \frac{3\ln(5)}{3\ln(2)+2\ln(5)}$$

Plugging this into a calculator correctly, we get $x = .9112919$. (Note: It may seem silly, but practice putting this into your calculator to make sure you get this answer! There are lots of parentheses that you have to be careful about, so it's worth making sure you know how to use your calculator to get the right answer!)

We can also use this technique to solve word problems involving the applications that we studied in the Section 4.2 notes.

Example 5. Suppose you put \$1000 into a bank account that compounds interest continuously. After 15 years, the balance of the account is \$1500. What was the annual interest rate?

Answer. Using our formula for continuously compounded interest, we have that $A(t) = Pe^{rt}$. Here, we know $P = 1000$, $t = 15$, and $A(15) = 1500$. That tells us that

$$1500 = 1000e^{15r}.$$

We isolate the exponential by dividing both sides by 1000:

$$1.5 = e^{15r}$$

Now, we can take ln of both sides and simplify:

$$\begin{aligned}\ln(1.5) &= \ln(e^{15r}) \\ &= 15r\ln(e) \\ &= 15r\end{aligned}$$

Dividing both sides by 15, we get $r = \dfrac{\ln(1.5)}{15} = .027$. So the interest rate was 2.7%.

Logarithmic Equations.
A logarithmic equation is one in which the variable appears in a logarithmic expression. Our general idea is again to use the fact that logarithms and exponential expressions undo each other. In this direction, this means that $a^{\log_a(x)} = x$.

Example 6. Solve the equation $\log_7(3x+1) = 2$.

Answer. We apply the exponential function of base 7 to both sides. In other words we raise 7 to the power of each side. (Note that this is NOT the same as raising each side to the 7th power! That's not what we want to do here!)
That is, we have

$$7^{\log_7(3x+1)} = 7^2$$

On the left, since the exponential function undoes the logarithm, we just get $3x+1$. On the right, we compute that $7^2 = 49$. Therefore, we have that $3x+1 = 49$. This is just a simple linear equation! By subtracting 1 from both sides and then dividing both sides by 3, we get $x = 16$.

While this last example was pretty straightforward, we can use the log laws to do fancier things. In particular, whenever we have an equation with multiple logs of the same base, we can always put them all on one side and combine them into a single log! From there, we can proceed in the same way as in the last example. However, there is an added wrinkle at the end! Since we know that we can't take a logarithm of a negative number or 0, we have to check our solution to make sure it works!

General Strategy for Solving Logarithmic Equations

(1) Put all the logarithm terms on one side, and all terms without a logarithm on the other side.

(2) Use the log laws to combine all the logarithm terms into a single logarithm.

(3) Raise the base to the power of each side. (So that each side of the equation from the last step appears in the exponent.)

(4) Use that the exponential undoes the logarithm to simplify.

(5) Solve the resulting equation for the variable.

(6) Check the solution to make sure it works in the original equation.

Example 7. Solve the equation $\log_2(x-3) = 2 - \log_2(x)$.

Answer. First, we put all the logs on the left.

$$\log_2(x-3) + \log_2(x) = 2$$

Next, we use log laws to combine on the right. The sum of logs is the log of the product, so we get that

$$\log_2((x-3)x) = 2$$

Then we can raise 2 to the power of each side.

$$2^{\log_2((x-3)x)} = 2^2$$

We simplify on each side.

$$(x-3)x = 4$$

We multiply out on the left.

$$x^2 - 3x = 4$$

But then this is a quadratic equation! We know how to solve this: we just put everything on one side and try to factor or use the quadratic formula. That is,

$$x^2 - 3x - 4 = 0 \quad \Rightarrow \quad (x-4)(x+1) = 0 \quad \Rightarrow \quad x = 4 \text{ or } x = -1$$

But we have to check our solutions! $x = 4$ is a valid solution, because the inputs in the logarithmic expressions, $x - 3$ and x, are both positive when we plug in $x = 4$. However, $x = -1$ is NOT a valid solution, because $\log_2(-1)$ is undefined! (Also, $\log_2(-1-3)$ is undefined too. As long as at least one of the expressions is undefined, the value of x cannot be a valid solution.)

Therefore, the only solution is $x = 4$.

SUMMARY:

- To solve exponential equations, we apply ln and use log laws to reduce to a simpler equation.

- To solve logarithmic equations, we use log laws to combine all logarithms into a single logarithm and then apply the exponential function with the same base to reduce to a simpler equation.

- We must check our answers when solving logarithmic equations!