Solution to Bonus Problem

The problem was as follows:
Consider the polar equation

$$r = \frac{ed}{1 + e\cos(\theta - \tan^{-1}(-\frac{1}{m}))}.$$

Put this into rectangular coordinates in terms of e, m, and d.

Solution: We can start by multiplying both sides by the denominator on the right:

$$r + re\cos\left(\theta - \tan^{-1}\left(-\frac{1}{m}\right)\right) = ed.$$

Next, we use the difference of angles formula for cosine:

$$r + re\left[\cos(\theta)\cos\left(\tan^{-1}\left(-\frac{1}{m}\right)\right) + \sin(\theta)\sin\left(\tan^{-1}\left(-\frac{1}{m}\right)\right)\right] = ed.$$

The next step is to get rid of the trig functions. We will use the fact that $x = r\cos(\theta)$ and $y = r\sin(\theta)$.
Then we also need to find algebraic formulas for $\cos\left(\tan^{-1}\left(-\frac{1}{m}\right)\right)$ and $\sin\left(\tan^{-1}\left(-\frac{1}{m}\right)\right)$. This is actually a tricky point, because the sign of each will change, depending on the sign of m. In other words, let's consider what happens to these compositions when the input changes signs. If $k > 0$, then $\tan^{-1}(k)$ will be an angle in quadrant I, so both $\sin(\tan^{-1}(k))$ and $\cos(\tan^{-1}(k))$ should be positive. If $k < 0$, then $\tan^{-1}(k)$ will be an angle in quadrant IV, so $\cos(\tan^{-1}(k))$ will be positive, but $\sin(\tan^{-1}(k))$ will be negative.
Let's start with $m > 0$, which makes the input negative. Then by using trig identities (or drawing out a triangle with opposite side -1 and adjacent side m), we find that

$$\cos\left(\tan^{-1}\left(-\frac{1}{m}\right)\right) = \frac{m}{\sqrt{m^2+1}}, \text{ and } \sin\left(\tan^{-1}\left(-\frac{1}{m}\right)\right) = -\frac{1}{\sqrt{m^2+1}}.$$

Next, if $m < 0$, the input is positive, so both cos and sin should be positive. We can adjust the signs to conclude that in this case

$$\cos\left(\tan^{-1}\left(-\frac{1}{m}\right)\right) = -\frac{m}{\sqrt{m^2+1}}, \text{ and } \sin\left(\tan^{-1}\left(-\frac{1}{m}\right)\right) = \frac{1}{\sqrt{m^2+1}}.$$

If we don't want to split into cases, we can use some trickery with absolute value. In particular, we want the cosine term to be positive no matter what, so we can replace m by $|m|$. The sine term is a little tougher, because we want it to be negative when m is positive, and positive when m is negative. It turns out that we can rig this by multiplying by $-\frac{m}{|m|}$. That is, we have the following formulas that work for all values of m:

$$\cos\left(\tan^{-1}\left(-\frac{1}{m}\right)\right) = \frac{|m|}{\sqrt{m^2+1}}, \text{ and } \sin\left(\tan^{-1}\left(-\frac{1}{m}\right)\right) = -\frac{m}{|m|\sqrt{m^2+1}}.$$

This reduces our equation to:

$$r + ex\left(\frac{|m|}{\sqrt{m^2+1}}\right) - ey\left(\frac{m}{|m|\sqrt{m^2+1}}\right) = ed,$$

so we can isolate r and we have that

$$r = e\left(d + \frac{ym}{|m|\sqrt{m^2+1}} - \frac{x\,|m|}{\sqrt{m^2+1}}\right).$$

Now we want to get rid of r, and we know that $r^2 = x^2 + y^2$. Thus, it may be tempting to replace r by $\sqrt{x^2+y^2}$. However, this does not allow for the possibility that r could be negative, so this equation may not define the entire conic. (You can verify that this is the case for the hyperbola by graphing this equation for a value of $e > 1$.

Therefore, we must square both sides to arrive at a valid equation:

$$x^2 + y^2 = e^2\left(d + \frac{ym}{|m|\sqrt{m^2+1}} - \frac{x\,|m|}{\sqrt{m^2+1}}\right)^2.$$

While this solution is acceptable, if we do the algebra required to square the right side, we can put our conic in the form $Ax^2+Bxy+Cy^2+Dx+Ey=F$, as follows:

$$\left(1 - \frac{e^2m^2}{m^2+1}\right)x^2 + \left(\frac{2e^2m}{m^2+1}\right)xy + \left(1 - \frac{e^2}{m^2+1}\right)y^2 + \left(\frac{2e^2\,|m|\,d}{\sqrt{m^2+1}}\right)x - \left(\frac{2e^2md}{|m|\sqrt{m^2+1}}\right)y = e^2d^2.$$

Now that we have the answer, what else could we do?

First, I should come clean about what conic this equation actually represents. I had said in class that it defines a conic with focus $(0,0)$, eccentricity e, and whose directrix is the line $y = mx + B$, where $d = \dfrac{B}{\sqrt{m^2+1}}$. This is true for $m < 0$, but if you graph the equations for the conic and the directrix together, you'll notice that their relationship isn't quite right when $m > 0$. We can fix this by using the same sort of absolute value trickery we used in the solution. So to be more precise, this equation defines a conic with focus $(0,0)$, eccentricity e, and directrix $y = mx - \dfrac{m}{|m|}d\sqrt{m^2+1}$.

Next, we could further generalize our equation by shifting the conic, which is achieved by replacing x with $(x-h)$ and y with $y-k$, which moves the focus at $(0,0)$ to the point (h,k). This yields:

$$\begin{aligned} e^2d^2 = {} & \left(1 - \frac{e^2m^2}{m^2+1}\right)(x-h)^2 + \left(1 - \frac{e^2}{m^2+1}\right)(y-k)^2 \\ & + \left(\frac{2e^2m}{m^2+1}\right)(x-h)(y-k) + \left(\frac{2e^2\,|m|\,d}{\sqrt{m^2+1}}\right)(x-h) - \left(\frac{2e^2md}{|m|\sqrt{m^2+1}}\right)(y-k). \end{aligned}$$

Finally, the topic that is really interesting to me is whether we can discern information about the conic directly from its standard form:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey = F.$$

We know that the value of B here determines the slant, so it's no surprise that we can use it to compute the slope of the directrix, depending on the sign of $\frac{B}{A-C}$. If $\frac{B}{A-C} > 0$, then we have the following formula:

$$\frac{-1}{m} = \tan((\tan^{-1}(\frac{B}{A-C}))/2)$$

and thus

$$m = \frac{\sin(\tan^{-1}(\frac{B}{A-C}))}{\cos(\tan^{-1}(\frac{B}{A-C})) - 1}.$$

If $\frac{B}{A-C} < 0$, then $\tan^{-1}$ will give us an angle in quadrant IV when we really want an angle in quadrant II. We make this adjustment by adding π, which yields the following formula:

$$\frac{-1}{m} = \tan((\tan^{-1}(\frac{B}{A-C}) + \pi)/2)$$

and thus

$$m = \frac{\sin(\tan^{-1}(\frac{B}{A-C}) + \pi)}{\cos(\tan^{-1}(\frac{B}{A-C}) + \pi) - 1}.$$

Next, to get more information, we can imagine rotating this conic to get it back to a form with no xy term, which would allow us to complete the squares and write our conic in the familiar forms we're used to. In particular, the amount we want to rotate by is θ, where θ is an angle between 0 and π such that $\tan(2\theta) = \frac{B}{A-C}$. Once we have this value of θ, we rewrite our equation in a new set of variables, $\bar{x}$ and $\bar{y}$.

In particular, we let $x = \bar{x}\cos(\theta) - \bar{y}\sin(\theta)$, and we let $y = \bar{x}\sin(\theta) + \bar{y}\cos(\theta)$. By making these substitutions, we rewrite our equation, and it will be symmetric on the $\bar{x}\bar{y}$ coordinate system. I'll try to do this generally. Let's assume that $\frac{B}{A-C} > 0$. The other case will give us similar results, but will be a little more of a headache.
Then:

$$\sin(\theta) = \sin(\tan^{-1}\left(\frac{B}{A-C}\right)/2) = \sqrt{\frac{1 - \cos(\tan^{-1}\left(\frac{B}{A-C}\right))}{2}} = \sqrt{\frac{1 - \frac{1}{\sqrt{1+(\frac{B}{A-C})^2}}}{2}}$$

and

$$\cos(\theta) = \cos(\tan^{-1}\left(\frac{B}{A-C}\right)/2) = \sqrt{\frac{1 + \cos(\tan^{-1}\left(\frac{B}{A-C}\right))}{2}} = \sqrt{\frac{1 + \frac{1}{\sqrt{1+(\frac{B}{A-C})^2}}}{2}}.$$

Simplifying slightly yields

$$\sin(\theta) = \sqrt{\frac{\sqrt{(A-C)^2 + B^2} - |A-C|}{2\sqrt{(A-C)^2 + B^2}}} \text{ and } \cos(\theta) = \sqrt{\frac{\sqrt{(A-C)^2 + B^2} + |A-C|}{2\sqrt{(A-C)^2 + B^2}}}$$

Next, we want to make the substitutions $x = \bar{x}\cos(\theta) - \bar{y}\sin(\theta)$ and $y = \bar{x}\sin(\theta) + \bar{y}\cos(\theta)$. Let's do some of the smaller steps first, so we are less overwhelmed when we put it all together.

First, we have that

$$x^2 = (\bar{x}\cos(\theta) - \bar{y}\sin(\theta))^2 = \bar{x}^2\cos^2(\theta) - 2\bar{x}\bar{y}\cos(\theta)\sin(\theta) + \bar{y}^2\sin^2(\theta),$$

and

$$y^2 = (\bar{x}\sin(\theta) + \bar{y}\cos(\theta))^2 = \bar{x}^2\sin^2(\theta) + 2\bar{x}\bar{y}\cos(\theta)\sin(\theta) + \bar{y}^2\cos^2(\theta),$$

and

$$\begin{aligned} xy &= (\bar{x}\cos(\theta) - \bar{y}\sin(\theta))(\bar{x}\sin(\theta) + \bar{y}\cos(\theta)) \\ &= \bar{x}^2\cos(\theta)\sin(\theta) + \bar{x}\bar{y}\cos^2(\theta) - \bar{x}\bar{y}\sin^2(\theta) - \bar{y}^2\cos(\theta)\sin(\theta). \end{aligned}$$

In light of these, it appears that we'll want to look at the formulas for $\cos^2(\theta)$, $\sin^2(\theta)$, and $\cos(\theta)\sin(\theta)$. We have that

$$\cos^2(\theta) = \frac{\sqrt{(A-C)^2 + B^2} + |A-C|}{2\sqrt{(A-C)^2 + B^2}}, \ \sin^2(\theta) = \frac{\sqrt{(A-C)^2 + B^2} - |A-C|}{2\sqrt{(A-C)^2 + B^2}},$$

and that

$$\begin{aligned} \cos(\theta)\sin(\theta) &= \sqrt{\left(\frac{\sqrt{(A-C)^2 + B^2} + |A-C|}{2\sqrt{(A-C)^2 + B^2}}\right)\left(\frac{\sqrt{(A-C)^2 + B^2} - |A-C|}{2\sqrt{(A-C)^2 + B^2}}\right)} \\ &= \sqrt{\frac{(\sqrt{(A-C)^2 + B^2} + |A-C|)(\sqrt{(A-C)^2 + B^2} - |A-C|)}{4((A-C)^2 + B^2)}} \\ &= \frac{|B|}{2\sqrt{(A-C)^2 + B^2}}. \end{aligned}$$

Now we take a deep breath and begin our substitution. We have that

$$\begin{aligned} F &= Ax^2 + Bxy + Cy^2 + Dx + Ey \\ &= A(\bar{x}^2\cos^2(\theta) - 2\bar{x}\bar{y}\cos(\theta)\sin(\theta) + \bar{y}^2\sin^2(\theta)) \\ &+ B(\bar{x}^2\cos(\theta)\sin(\theta) + \bar{x}\bar{y}\cos^2(\theta) - \bar{x}\bar{y}\sin^2(\theta) - \bar{y}^2\cos(\theta)\sin(\theta)) \\ &+ C(\bar{x}^2\sin^2(\theta) + 2\bar{x}\bar{y}\cos(\theta)\sin(\theta) + \bar{y}^2\cos^2(\theta)) \\ &+ D(\bar{x}\cos(\theta) - \bar{y}\sin(\theta)) + E(\bar{x}\sin(\theta) + \bar{y}\cos(\theta)) \end{aligned}$$

The most important thing to notice here is the $\bar{x}\bar{y}$ term, which we combine to write as

$$\left[B(\cos^2(\theta) - \sin^2(\theta)) + 2\cos(\theta)\sin(\theta)(C-A)\right]\bar{x}\bar{y}.$$

Substituting in the values for sin and cos makes this coefficient reduce as

$$\frac{B\,|A-C|}{\sqrt{(A-C)^2 + B^2}} + \frac{|B|\,(C-A)}{\sqrt{(A-C)^2 + B^2}},$$

and since $\dfrac{B}{A-C} > 0$, this tells us that B and $(C-A)$ have opposite signs, so this coefficient becomes 0.

Unfortunately, the other terms aren't as nice. The other quadratic terms,

$$\left[A\cos^2(\theta) + B\cos(\theta)\sin(\theta) + C\sin^2(\theta)\right]\bar{x}^2$$

and

$$\left[A\sin^2(\theta) - B\cos(\theta)\sin(\theta) + C\cos^2(\theta)\right]\bar{y}^2$$

do not simplify as nicely. However, we can still go ahead and complete the square. Let's take care of the $\bar{x}$ terms first. To save space in displaying these terms, I'll let $\bar{A} = A\cos^2(\theta) + B\cos(\theta)\sin(\theta) + C\sin^2(\theta)$. Then

$$\begin{aligned}
&\left[A\cos^2(\theta) + B\cos(\theta)\sin(\theta) + C\sin^2(\theta)\right]\bar{x}^2 + \left[D\cos(\theta) + E\sin(\theta)\right]\bar{x} \\
&= \bar{A}\left(\bar{x}^2 + \left(\frac{D\cos(\theta) + E\sin(\theta)}{\bar{A}}\right)\bar{x}\right) \\
&= \bar{A}\left(\bar{x}^2 + \left(\frac{D\cos(\theta) + E\sin(\theta)}{\bar{A}}\right)\bar{x} + \frac{(D\cos(\theta) + E\sin(\theta))^2}{4\bar{A}^2} - \frac{(D\cos(\theta) + E\sin(\theta))^2}{4\bar{A}^2}\right) \\
&= \bar{A}\left(\left(\bar{x} + \frac{D\cos(\theta) + E\sin(\theta)}{2\bar{A}}\right)^2 - \frac{(D\cos(\theta) + E\sin(\theta))^2}{4\bar{A}^2}\right) \\
&= \bar{A}\left(\bar{x} + \frac{D\cos(\theta) + E\sin(\theta)}{2\bar{A}}\right)^2 - \frac{(D\cos(\theta) + E\sin(\theta))^2}{4\bar{A}}.
\end{aligned}$$

The $\bar{y}$ terms work similarly, and I'll let $\bar{C} = A\sin^2(\theta) - B\cos(\theta)\sin(\theta) + C\cos^2(\theta)$. Then

$$\begin{aligned}
&\left[A\sin^2(\theta) - B\cos(\theta)\sin(\theta) + C\cos^2(\theta)\right]\bar{y}^2 + \left[E\cos(\theta) - D\sin(\theta)\right]\bar{y} \\
&= \bar{C}\left(\bar{y}^2 + \left(\frac{E\cos(\theta) - D\sin(\theta)}{\bar{C}}\right)\bar{y}\right) \\
&= \bar{C}\left(\bar{y}^2 + \left(\frac{E\cos(\theta) - D\sin(\theta)}{\bar{C}}\right)\bar{y} + \frac{(E\cos(\theta) - D\sin(\theta))^2}{4\bar{C}^2} - \frac{(E\cos(\theta) - D\sin(\theta))^2}{4\bar{C}^2}\right) \\
&= \bar{C}\left(\left(\bar{y} + \frac{E\cos(\theta) - D\sin(\theta)}{2\bar{C}}\right)^2 - \frac{(E\cos(\theta) - D\sin(\theta))^2}{4\bar{C}^2}\right) \\
&= \bar{C}\left(\bar{y} + \frac{E\cos(\theta) - D\sin(\theta)}{2\bar{C}}\right)^2 - \frac{(E\cos(\theta) - D\sin(\theta))^2}{4\bar{C}}.
\end{aligned}$$

So our original equation for a conic in standard form has been reduced to:

$$\begin{aligned}
&\bar{A}\left(\bar{x} + \frac{D\cos(\theta) + E\sin(\theta)}{2\bar{A}}\right)^2 + \bar{C}\left(\bar{y} + \frac{E\cos(\theta) - D\sin(\theta)}{2\bar{C}}\right)^2 \\
&= F + \frac{(D\cos(\theta) + E\sin(\theta))^2}{4\bar{A}} + \frac{(E\cos(\theta) - D\sin(\theta))^2}{4\bar{C}}
\end{aligned}$$

At this point, we can start to diagnose the type of conic. If either $\bar{A}$ or $\bar{C}$ is 0, the conic will be a parabola. If $\bar{A}$ and $\bar{C}$ have different signs, the conic will be a hyperbola. If $\bar{A}$ and $\bar{C}$ have the same sign, the conic will be an ellipse, if it exists. Note the possibility that the conic does not exist, if, for example, $\bar{A}$ and $\bar{C}$ are both positive and the right side of the equation is negative. All of these things can be determined by the sign of the product $\bar{A}\bar{C}$. So now we'll calculate this product, $\bar{A}\bar{C}$. The algebra here is messy, but if we keep our wits about us, we'll be rewarded with something wonderful.

$$
\begin{aligned}
\bar{A}\bar{C} &= (A\cos^2(\theta) + B\cos(\theta)\sin(\theta) + C\sin^2(\theta))(A\sin^2(\theta) - B\cos(\theta)\sin(\theta) + C\cos^2(\theta)) \\
&= A^2\cos^2(\theta)\sin^2(\theta) - AB\cos^3(\theta)\sin(\theta) + AC\cos^4(\theta) + AB\cos(\theta)\sin^3(\theta) - B^2\cos^2(\theta)\sin^2(\theta) \\
&\qquad + BC\cos^3(\theta)\sin(\theta) + AC\sin^4(\theta) - BC\cos(\theta)\sin^3(\theta) + C^2\cos^2(\theta)\sin^2(\theta) \\
&= (A^2 - B^2 + C^2)\cos^2(\theta)\sin^2(\theta) - B(A-C)\cos(\theta)\sin(\theta)(\cos^2(\theta) - \sin^2(\theta)) \\
&\qquad + AC(\cos^4(\theta) + \sin^4(\theta))
\end{aligned}
$$

Then we use our formulas for $\sin(\theta)$ and $\cos(\theta)$ as needed, and plug in.

$$
\begin{aligned}
&= \frac{(A^2 - B^2 + C^2)B^2}{4((A-C)^2 + B^2)} - \frac{2B(A-C)\,|B|\,|A-C|}{4((A-C)^2 + B^2)} + \frac{2AC(2(A-C)^2 + B^2)}{4((A-C)^2 + B^2)} \\
&= \frac{A^2B^2 - B^4 + B^2C^2 - 2B^2(A-C)^2 + 4AC(A-C)^2 + 2ACB^2}{4((A-C)^2 + B^2)} \\
&= \frac{A^2B^2 - B^4 + B^2C^2 - 2A^2B^2 + 4ACB^2 - 2B^2C^2 + 4AC(A-C)^2 + 2ACB^2}{4((A-C)^2 + B^2)} \\
&= \frac{4AC(A-C)^2 + B^2\left[-A^2 + 2AC - C^2 + 4AC - B^2\right]}{4((A-C)^2 + B^2)} \\
&= \frac{4AC(A-C)^2 + B^2\left[-(A-C)^2 + 4AC - B^2\right]}{4((A-C)^2 + B^2)} \\
&= \frac{4AC(A-C)^2 - B^2(A-C)^2 + B^2\left[4AC - B^2\right]}{4((A-C)^2 + B^2)} \\
&= \frac{(A-C)^2\left[4AC - B^2\right] + B^2\left[4AC - B^2\right]}{4((A-C)^2 + B^2)} \\
&= \frac{\left[(A-C)^2 + B^2\right]\left[4AC - B^2\right]}{4((A-C)^2 + B^2)} \\
&= \frac{4AC - B^2}{4}.
\end{aligned}
$$

In other words, $-4\bar{A}\bar{C} = B^2 - 4AC$. We have thus found the following theorem.

Theorem 1. Suppose the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey = F$ defines a nondegenerate conic. Then:

(1) If $B^2 - 4AC > 0$, the conic is a hyperbola.
(2) If $B^2 - 4AC = 0$, the conic is a parabola.
(3) If $B^2 - 4AC < 0$, the conic is an ellipse.

This is a pretty remarkable result! However, I have my sights set even higher: from the coefficients, can we get a formula for the exact eccentricity? Stay tuned!