Tools for establishing convergence/divergence of series

(1) Express the given series as a geometric series, p-series, or telescoping series.

(2) (Test for divergence) If the statement "$\lim_{n\to\infty} a_n = 0$" is false, then $\sum a_n$ diverges.

(3) (Algebra of limits) If $\sum a_n$ and $\sum b_n$ converge and c is a real number, then $\sum(a_n + b_n)$ and $\sum ca_n$ converge, and

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \text{ and } \sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n.$$

(4) (Monotone sequence theorem with bounded partial sums) Suppose $a_n \geq 0$ for all n. The series $\sum a_n$ is convergent if and only if the sequence of partial sums (s_n) is bounded.

(5) (Integral test) Suppose $a_n \geq 0$ for all n and suppose f is a continuous, decreasing function on $[b, \infty)$ for some $b > 0$, such that $f(n) = a_n$ for each n. Then the series $\sum a_n$ and the improper integral $\int_b^\infty f(x)\,dx$ either both converge or both diverge.

(6) (Comparison test) Suppose $a_n \geq 0$ for all n.
- If $0 \leq a_n \leq b_n$ for all n and $\sum b_n$ converges, then $\sum a_n$ converges.
- If $0 \leq b_n \leq a_n$ for all n and $\sum b_n$ diverges, then $\sum a_n$ diverges.

(7) (Limit comparison test) Suppose $a_n \geq 0$ and $b_n > 0$ for all n. If $\lim_{n\to\infty} \frac{a_n}{b_n}$ is nonzero and finite, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

(8) (Alternating series test) If $0 \leq a_{n+1} \leq a_n$ for all n and $\lim_{n\to\infty} a_n = 0$, then the alternating series $\sum(-1)^{n-1}a_n$ converges.

(9) (Ratio test) Suppose $a_n \neq 0$ for all n.
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum a_n$ is absolutely convergent.
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum a_n$ is divergent.
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then no conclusions can be drawn.

(10) (Root test)
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then $\sum a_n$ is absolutely convergent.
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$, then $\sum a_n$ is divergent.
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then no conclusions can be drawn.