Math 251: Multivariable Calculus **Name (Print):** ____________________
Spring 2022
Practice Problems for Final Exam

---

1. In 3D space, does the equation $x^2 + y^2 + z^2 = 4$ describe a line, circle, plane, cylinder, or sphere?

2. In 3D space, does the equation $x^2 + z^2 = 9$ describe a line, circle, plane, cylinder, or sphere?

3. In 3D space, does the equation $4x+y-z = 6$ describe a line, circle, plane, cylinder, or sphere?

4. Suppose $\mathbf{v} = \langle 2, -1, 6\rangle$ and $\mathbf{w} = \langle -1, -1, 2\rangle$ are vectors. Find $\mathbf{v}+\mathbf{w}$, find $\mathbf{v}\cdot\mathbf{w}$, find $\mathbf{v}\times\mathbf{w}$, find $|\mathbf{v}|$ and $|\mathbf{w}|$, find the angle between the vectors $\mathbf{v}$ and $\mathbf{w}$, and determine if they are parallel, perpendicular, or neither.

5. Find parametric equations and symmetric equations for the line through the points $(-8, 1, 4)$ and $(3, -2, 4)$.

6. Find parametric equations and symmetric equations for the line of intersection between the planes $x + 2y + 3z = 1$ and $x - y + z = 1$.

7. Find an equation for the plane though the point $(5, 3, 5)$ and with normal vector $\langle 2, 1, -1\rangle$.

8. Find an equation for the plane through the point $(1, -1, -1)$ and parallel to the plane $5x - y - z = 6$.

9. Find an equation for the plane through $(6, -1, 3)$ that contains the line with symmetric equations $\dfrac{x}{3} = y + 4 = \dfrac{z}{2}$.

10. Determine whether the planes $x - y + 3z = 1$ and $3x + y - z = 2$ are parallel, perpendicular, or neither.

11. Find the domain of the vector function $\mathbf{r}(t) = \left\langle \cos(t), \ln(t), \frac{1}{t-2} \right\rangle$.

12. Find the limit $\lim_{t\to\infty} \left\langle te^{-t}, \dfrac{t^3+t}{2t^3-1}, t\sin\left(\dfrac{1}{t}\right) \right\rangle$.

13. Write a vector equation and parametric equations for the line segment connecting the point $(2, 0, 0)$ to the point $(6, 2, -2)$.

14. Consider a vector function $\mathbf{r}(t) = \langle t^2, \frac{2}{3}t^3, t\rangle$. Find $\mathbf{r}'(t)$, the unit tangent vector $\mathbf{T}(t)$, the unit normal vector $\mathbf{N}(t)$, the binormal vector $\mathbf{B}(t)$, the curvature $\kappa(t)$, the definite integral $\int_0^1 \mathbf{r}(t)\,dt$, and write an integral that gives the arc length of the curve drawn out by $\mathbf{r}(t)$ from $t = 0$ to $t = 1$.

15. A rifle is fired from sea level with angle of elevation $36°$. What is the initial speed of the bullet if the maximum height is 1600 ft?

16. If a particle has position function $\mathbf{r}(t) = t\mathbf{i} + 2\cos(t)\mathbf{j} + \sin(t)\mathbf{k}$, find the velocity, acceleration, and speed functions of the particle.

17. Determine the domain of the function $f(x, y) = \dfrac{1}{1 - x^2 - y^2}$ and sketch this region in the xy-plane.

18. Compute $\lim_{(x,y)\to(1,-1)} \dfrac{x^2y + xy^2}{x^2 - y^2}$.

19. Show that $\lim_{(x,y)\to(0,0)} \dfrac{xy^2\cos(y)}{x^2 + y^4}$ does not exist.

20. Find an equation of the tangent plane to the surface $z = (x + 2)^2 - 2(y - 1)^2 - 5$ at the point $(2, 3, 3)$.

21. Find an equation of the tangent plane to the surface $z = x\sin(x + y)$ at the point $(-1, 1, 0)$.

22. Let $f(x,y) = x^2e^y$. Find $f_x(x,y)$, $f_y(x,y)$, $f_{xy}(x,y)$, $f_{yx}(x,y)$, $f_{xx}(x,y)$, $f_{yy}(x,y)$, $\nabla f(x,y)$, and the directional derivative $D_{\mathbf{u}}f(x,y)$ in the direction $\mathbf{u} = \langle 2, -1\rangle$.

23. Let $f(x,y,z) = x\ln(yz)$. Find $\dfrac{\partial f}{\partial x}(x,y,z)$, $\dfrac{\partial f}{\partial y}(x,y,z)$, $\dfrac{\partial f}{\partial z}(x,y,z)$, $\dfrac{\partial^2 f}{\partial x\partial y}(x,y,z)$, $\dfrac{\partial^2 f}{\partial y^2}(x,y,z)$, and $\dfrac{\partial^2 f}{\partial z\partial y}(x,y,z)$.

24. Let $f(x,y,z) = e^{xy^2+z^2}$. Find $\nabla f(x,y)$.

25. Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$, where $z = \sqrt{x}\, e^{xy}$, and $x = 1+st$, $y = s^2 - t^2$.

26. Find all local maxima, local minima, and saddle points of the function $f(x,y) = x^2+y^4+2xy$.

27. Find all local maxima, local minima, and saddle points of the function $f(x,y) = x^3-3x+3xy^2$.

28. Find three positive numbers whose sum is 12 and and the sum of whose squares is as small as possible.

29. Use Lagrange multipliers to find the extreme values of $f(x,y) = xy$ subject to the constraint $4x^2 + y^2 = 8$.

30. Use Lagrange multipliers to find the extreme values of $f(x,y) = x^2 + y^2 + 4x - 4y$ subject to the constraint $x^2 + y^2 = 9$.

31. Calculate $\int_1^2 \int_0^2 y + 2xe^y \, dx \, dy$.

32. Calculate $\int_0^1 \int_x^{e^x} 3xy^2 \, dy \, dx$.

33. Calculate $\int_0^{\pi} \int_0^1 \int_0^{\sqrt{1-y^2}} y \sin(x) \, dx \, dy \, dz$.

34. Calculate the integral $\int_0^1 \int_x^1 \cos(y^2) \, dy \, dx$ by first reversing the order of integration.

35. Rewrite the integral $\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} f(x, y, z) \, dz \, dy \, dx$ as an integral in the order $dx \, dy \, dz$.

36. Rewrite the integral $\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x^3 + xy^2 \, dy \, dx$ as an integral in polar coordinates, and then compute the integral.

37. Consider the solid region E that lies above the plane $z = 0$, below the plane $z = y$, and inside the cylinder $x^2 + y^2 = 4$. Compute $\iiint_E yz \, dV$.

38. Use the transformation $u = x - y$, $v = x + y$ to evaluate $\iint_R \frac{x - y}{x + y} \, dA$, where R is the square with vertices $(0, 2)$, $(1, 1)$, $(2, 2)$, and $(1, 3)$. (Hint: this transformation maps the rectangle S in the uv-plane onto R, where S has vertices $(-2, 2)$, $(0, 2)$, $(0, 4)$, and $(-2, 4)$.)

39. Suppose a lamina occupies the part of the disc $x^2 + y^2 \leq 16$ that lies in the first quadrant and has density function $\rho(x, y) = xy^2$. Compute the total mass and the center of mass of the lamina.

40. Use spherical coordinates to evaluate

$$\int_{-2}^2 \int_0^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2 \sqrt{x^2 + y^2 + z^2} \, dz \, dx \, dy.$$

41. Evaluate the line integral $\int_C x\,ds$, where C is the arc of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$, using the parametrization $x = t$, $y = t^2$.

42. Evaluate the line integral $\int_C yz\cos(x)\,ds$, where C is parametrized by $x = t$, $y = 3\cos(t)$, $z = 3\sin(t)$, for $0 \le t \le \pi$.

43. Evaluate the line integral $\int_C \sqrt{xy}\,dx + e^y\,dy + xz\,dz$, where C is drawn out by the vector function $\mathbf{r}(t) = \left\langle t^4,\, t^2,\, t^3 \right\rangle$, $0 \le t \le 1$.

44. Consider the vector field $\mathbf{F}(x,y) = \left\langle (1+xy)e^{xy},\, e^y + x^2e^{xy} \right\rangle$. Show that $\mathbf{F}$ is a conservative vector field, and find a function f such that $\mathbf{F} = \nabla f$.

45. Consider the vector field $\mathbf{F}(x,y) = \left\langle 4x^3y^2 - 2xy^3,\, 2x^4y - 3x^2y^2 + 4y^3 \right\rangle$. Show that $\mathbf{F}$ is a conservative vector field, and use this to evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the curve drawn out by the vector function $\mathbf{r}(t) = \langle t + \sin(\pi t),\, 2t + \cos(\pi t)\rangle$, $0 \le t \le 1$.

46. Suppose that C is the closed curve consisting of the portion of the parabola $y = x^2$ from $(-1,1)$ to $(1,1)$ and the line segment from $(1,1)$ to $(-1,1)$. Compute $\oint_C xy^2\,dx - x^2y\,dy$.

47. Use Green's Theorem to compute $\oint_C x^2y\,dx - xy^2\,dy$, where C is the circle $x^2 + y^2 = 4$, positively oriented.

48. Consider the vector field $\mathbf{F}(x,y) = \langle y - \cos(y),\, x\sin(y)\rangle$. Compute $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where C is the circle $(x-3)^2 + (y+4)^2 = 4$ traversed clockwise.

49. Find the work done by the force $\mathbf{F}(x,y) = \left\langle x(x+y),\, xy^2 \right\rangle$ in moving a particle from the origin along the x-axis to $(1,0)$, then along the line segment to $(0,1)$, and then back to the origin along the y-axis.

50. Consider the function $f(x,y) = xye^y - y^2\cos(e^{xy})$. Compute $\oint_C \nabla f\cdot d\mathbf{r}$, where C is the ellipse $6x^2 + 7y^2 = 30$ traversed counterclockwise.