Math 261: Differential Equations **Name (Print):** ______________________

Fall 2022

Practice Problems for Final Exam

Find the general solution of each of the following first-order differential equations by identifying them as separable, linear, exact, or homogeneous (in the first-order non-linear sense), and then applying the appropriate solution method.

1. $\dfrac{dy}{dx} = e^{3x+2y}$

2. $(x+y)^2\,dx + (2xy + x^2 - 1)\,dy = 0$

3. $x\dfrac{dy}{dx} - y = x^2\sin(x)$

4. $\dfrac{dy}{dx} = 4(y^2+1)$

5. $y' - \sin(x)\,y = 2\sin(x)$

6. $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$

7. $(x^2 - y^2)\,dx + (x^2 - 2xy)\,dy = 0$

8. $y\,dx - 4(x + y^6)\,dy = 0$

9. $(\tan(x) - \sin(x)\sin(y))\,dx + \cos(x)\cos(y)\,dy = 0$

10. $(x-y)\,dx + x\,dy = 0$

11. If possible, find the particular solution to each of the equations on the last page subject to initial conditions:

    (a) $y(0) = 0$
    (b) $y(1) = 0$
    (c) $y(0) = 1$

12. The equation $(-xy\sin(x)+2y\cos(x))dx+2x\cos(x)dy = 0$ is not exact. Show that $\mu(x, y) = xy$ is an integrating factor that makes the equation exact, and then solve.

13. The population of a town grows at a rate proportional to the population present at time t. The initial population of 500 increases by 15% in 10 years. Find the function $P(t)$ giving the population at time t.

14. A thermometer reading 70° F is placed in an oven preheated to a constant temperature. Through a glass window in the oven door, an observer records that the thermometer reads 110° F after $\frac{1}{2}$ minute and 145° F after 1 minute. How hot is the oven?

15. A large tank is filled to capacity with 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped into the tank at a rate of a 5 gal/min. The well-mixed solution is pumped out at the same rate. Find the number $A(t)$ of pounds of salt in the tank at time t.

16. Calculate the Wronskian of the functions $f_1(x) = x^2$ and $f_2(x) = 4x - 3x^2$ and use it to determine whether they are linearly independent on $(-\infty, \infty)$.

17. Calculate the Wronskian of the functions $f_1(x) = e^x \sin(2x)$ and $f_2(x) = e^x \cos(2x)$ and use it to determine whether they are linearly independent on $(-\infty, \infty)$.

18. Given that $y_1 = \ln(x)$ is one solution to $xy'' + y' = 0$, use the reduction of order formula to find a second solution that is linearly independent to y_1.

19. Given that $y_1 = x+1$ is one solution to $(1-2x-x^2)y''+2(1+x)y'-2y = 0$, use the reduction of order formula to find a second solution that is linearly independent to y_1.

Give the general solution of each of the following differential equations.

20. $y'' - y' - 6y = 0$

21. $y'' + 8y' + 16y = 0$

22. $y'' + 9y = 0$

23. $3y'' + 2y' + y = 0$

24. $y'' - 10y' + 25y = 0$

25. $\frac{1}{4}y'' + y' + y = x^2 - 2x$

26. $y'' + 4y = 3\sin(2x)$

27. $y'' + 2y' + y = \sin(x) + 3\cos(2x)$

28. $5y'' + y' = -6x$

29. $y'' + 4y' + 5y = 35e^{-4x}$

30. If possible, find the particular solution to each of the equations on this page subject to initial conditions:

    (a) $y(0) = 0$, $y'(0) = 0$
    (b) $y(0) = 0$, $y'(0) = 1$
    (c) $y(0) = 1$, $y'(0) = 0$

31. Use variation of parameters to solve $y'' + y = \sec(x)$.

32. Use variation of parameters to solve $y'' + y = \cos^2(x)$.

33. Use variation of parameters to solve $y'' - 4y = \dfrac{e^{2x}}{x}$.

34. Use variation of parameters to solve $y'' + 2y' + y = e^{-x}\ln(x)$.

35. Solve the equation $x^2y'' - 3xy' - 2y = 0$.

36. Solve the equation $3x^2y'' + 6xy' + y = 0$.

37. Solve the equation $xy'' + y' = x$.

38. A mass of $\frac{1}{8}$ kilograms is attached to a spring with spring constant 16 N/m and released at rest from a point 10 cm below the equilibrium. If no damping occurs, find the equation of motion and determine the period.

39. A mass of 20 kilograms is attached to a spring with spring constant 20 N/m. The mass is released from the equilibrium position with an upward velocity of 10 m/s. Find the equation of motion if no damping occurs.

40. A 1-kilogram mass is attached to a spring with spring constant 16 N/m, and damping occurs with damping coefficient $\beta = 10$. Determine the equation of motion if the mass is initially released from rest from a point 1 meter below the equilibrium, and determine whether the system is underdamped, overdamped, or critically damped.

41. A mass of $\frac{1}{8}$ kilograms stretches a spring 1.6 meters. Find the value of the damping constant β so that the system is critically damped.

42. A mass of 2 kilograms is attached to a spring with spring constant 32 N/m, and is allowed to come to equilibrium. Starting at time $t = 0$, a force of $f(t) = 68e^{-2t}\cos(4t)$ is applied to the system. Find the equation of motion in the absence of damping.

43. Use the definition of the Laplace transform to write an integral giving $\mathscr{L}\{\cos^3(t)\}$. You do not need to solve this integral.

44. Compute $\mathscr{L}^{-1}\left\{\dfrac{1}{4s+1}\right\}$.

45. Compute $\mathscr{L}^{-1}\left\{\dfrac{2s-6}{s^2+9}\right\}$.

46. Compute $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+2s-3}\right\}$.

47. Compute $\mathscr{L}^{-1}\left\{\dfrac{1}{s^3+5s}\right\}$.

48. Compute $\mathscr{L}^{-1}\left\{\dfrac{1}{(s+2)^3}\right\}$.

49. Compute $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+4s+5}\right\}$.

50. Compute $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s^3}\right\}$.

51. Compute $\mathscr{L}\{(t-1)\mathscr{U}(t-1)\}$.

52. Compute $\mathscr{L}\{t\cos(2t)\}$.

53. Compute $\mathscr{L}\left\{\displaystyle\int_0^t e^{-\tau}\cos(\tau)\,d\tau\right\}$.

54. Consider the piecewise defined function $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ t^2 & \text{if } t \ge 1 \end{cases}$. Write f in terms of unit step functions and compute $\mathscr{L}\{f(t)\}$.

Use the Laplace transform to solve each initial-value problem.

55. $y' + 6y = e^{4t}$, $y(0) = 2$

56. $y'' + y = \sqrt{2}\sin(\sqrt{2}t)$, $y(0) = 10$, $y'(0) = 0$

57. $y'' - y' = e^t \cos(t)$, $y(0) = 0$, $y'(0) = 0$

58. $y'' + 4y = \sin(t)\mathscr{U}(t - 2\pi)$, $y(0) = 1$, $y'(0) = 0$

59. $y'' + 16y = f(t)$, $y(0) = 0$, $y'(0) = 1$, where $f(t) = \begin{cases} \cos(4t) & \text{if } 0 \leq t < \pi \\ 0 & \text{if } t \geq \pi \end{cases}$

60. $y'(t) = 1 - \sin(t) - \displaystyle\int_0^t y(\tau)\,d\tau$, $y(0) = 0$

61. $y(t) + \displaystyle\int_0^t (t - \tau)y(\tau)\,d\tau = t$

62. $y(t) = te^t + \displaystyle\int_0^t \tau y(t - \tau)\,d\tau$

63. Rewrite the expression using a single power series whose general term involves x^k:

$$\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n.$$

64. Rewrite the expression using a single power series whose general term involves x^k:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2\sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^n.$$

Find two power series solutions of each differential equation about the ordinary point $x = 0$.

65. $y'' + x^2y' + xy = 0$

66. $(x^2 + 2)y'' + 3xy' - y = 0$

67. $(x - 1)y'' - xy' + y = 0$

68. For each of the following equations, determine whether 0 is an ordinary point, a regular singular point, or an irregular singular point.
    (a) $2xy'' - y' + 2y = 0$
    (b) $x^2y'' + (3x - 1)y' + y = 0$
    (c) $(x - 1)y'' + y' = 0$

69. Each of the following equations has a regular singular point at $x = 0$. Find the indicial roots of each equation, and determine how many power series solutions can be expected.
    (a) $4xy'' + \frac{1}{2}y' + y = 0$
    (b) $xy'' - xy' + y = 0$
    (c) $x^2y'' + (\frac{5}{3}x + x^2)y' - \frac{1}{3}y = 0$

70. Find two power series solutions to $9x^2y'' + 9x^2y' + 2y = 0$.

71. Find one power series solution to $xy'' + (1 - x)y' - y = 0$. Do not attempt to find a second power series solution.

72. Show that the functions $f_1(x) = x$ and $f_2(x) = \cos(2x)$ are orthogonal on $[-\pi/2, \pi/2]$.

73. Find the Fourier series of the function $f(x) = \begin{cases} 0, & -2 < x < -1 \\ -2, & -1 \leq x < 0 \\ 1, & 0 \leq x < 1 \\ 0, & 1 \leq x < 2 \end{cases}$,

and find the number to which the series converges at each discontinuity.

Find the Fourier series of each function.

74. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 \le x < \pi \end{cases}$

75. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin(x), & 0 \le x < \pi \end{cases}$

76. $f(x) = e^x$, $\pi < x < \pi$

77. $f(x) = \begin{cases} 1, & -5 < x < 0 \\ 1 + x, & 0 \le x < 5 \end{cases}$

78. Find the Fourier sine series of $f(x) = \begin{cases} x - 1, & -\pi < x < 0 \\ x + 1, & 0 \le x < \pi \end{cases}$.

79. Find the Fourier cosine series of $f(x) = x^2$, $-1 < x < 1$.

80. Find the Fourier cosine series of $f(x) = \begin{cases} -\sin(x), & -\pi < x < 0 \\ \sin(x), & 0 \le x < \pi \end{cases}$.

81. Find the Fourier sine series of $f(x) = \begin{cases} -\cos(x), & -\pi < x < 0 \\ \cos(x), & 0 \le x < \pi \end{cases}$.

82. Find the half-range cosine and sine expansions of $f(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 \le x < 2 \end{cases}$.

83. Find the half-range cosine and sine expansions of $f(x) = x^2 + x$, $0 < x < 1$.

84. Find the half-range cosine and sine expansions of $f(x) = x + 1$, $0 < x < 1$.

85. Find a Fourier series solution to the equation $y'' + 10y = f(x)$, where

$$f(x) = \begin{cases} 5, & 0 < x < \pi \\ -5, & \pi \le x < 2\pi \end{cases}$$

For each partial differential equation, use separation of variables to find, if possible, product solutions.

86. $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y}$

87. $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = u$

88. $x\dfrac{\partial u}{\partial x} = y\dfrac{\partial u}{\partial y}$

89. $k\dfrac{\partial^2 u}{\partial x^2} - u = \dfrac{\partial u}{\partial t}$, for $k > 0$

90. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = u$