MATH 448 **Name (Print):** ____________________
Spring 2020
Practice Problems

These problems are practice for the final exam. This is not intended to be an accurate representation of the length of the exam. In addition to these problems, it will be useful to review all homework problems from the course. Please note the following instructions:

You may *not* use any books, or calculator on the final exam. However, you will be permitted to create and use a single 8.5 inch by 11 inch sheet of paper with any notes you'd like. Additionally, on the final exam, you will be given a chart of probabilities of the standard normal distribution.

As usual, unless otherwise stated in the problem, you may leave all answers in terms of $\binom{n}{k}$, $P(n, k)$, $k!$, e, log, or any sum, difference, product, or quotient of such symbols.

Partial credit will only be awarded to answers for which an explanation and/or work is shown.

1. A math professor decides to assign grades using some continuous random variable, but doesn't know which one to use. He gives the students four different options for them to vote on. The options are:

   1. Grades will be normally distributed with mean 76 and standard deviation 6.
   2. Grades will be uniformly distributed on $(50, 100)$.
   3. Grades will be exponentially distributed with parameter $\lambda = \frac{1}{85}$.
   4. Grades will be $100X$, where X has the following cumulative distribution function:

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{7}{4}x^3 - \frac{3}{4}x^7 & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$$

   (a) Alice only cares about getting an A, so she will vote for the option that gives the highest probability of getting a score of at least 90. Which option will she choose?

   (b) Bob wants to get at least a B, so he will vote for the option that gives the highest probability of getting a score of at least 80. Which option will he choose?

   (c) Patricia just needs to pass, so she will vote for the option that gives the highest probability of getting a score of 70 or better. Which option will she choose?

   (d) Nelia hates uncertainty, so she will vote for the option with the smallest variance, so that she can better predict what her grade will be. Which option will she choose?

   (e) Fred doesn't really understand what's going on, so he will vote for the option with the highest expected value. Which option will he choose?

2. (20 points) (a) A Pennsylvania license plate has 3 letters followed by 4 digits (ex: AXH-6259). You are assigned a random Pennsylvania license plate. What is the probability that all 3 letters in the license plate are the same? (ex: JJJ-1764)

(b) You flip a fair coin 4 times. Find the probability of getting exactly 3 H given that at least one flip was H and at least one flip was T.

(c) Recall that a standard deck of 52 cards has 26 red cards and 26 black cards. You draw 13 cards at random without replacement. What is the probability that all 13 cards are red?

(d) Suppose that A and B are events in some sample space, and assume that $P(A^c) = .3$, $P(B) = .5$, and $P(A^c \cap B) = .4$. Are A and B independent?

3. (20 points) Suppose a computer generates random numbers one at a time according to a uniform distribution on $(0, 5)$, with all generated numbers being independent of each other.

   (a) What is the probability that the first number generated is less than 1?

   (b) What is the probability that none of the first 10 numbers generated are less than 1?

   (c) Suppose the computer generates 100 numbers. What is the expected number of them that are less than 1?

4. (20 points) A waiter notices that the amounts of time that customers spend at a particular table are exponentially distributed with mean 1 hour.

   (a) What is the probability that a customer stays at the table for over 2 hours?

   (b) Suppose that a customer has already been at the table for an hour and a half. What is the probability that she will stay at the table for an additional hour?

   (c) On a certain day, 5 different customers use the table, with the amounts of time spent at the table all independent of each other. Let Y be the minimum of the amounts of time at the table of the 5 customers. What is the probability that Y is at least $1/2$ hour?

5. (20 points) As in the previous problem, the amounts of time that customers spend at a particular table are exponentially distributed with mean 1 hour. A busboy at the restaurant is responsible for clearing the dishes from the table when customers leave. On a busy day, whenever a customer leaves, a new customer instantly claims the table, and the busboy must quickly clear the table. Suppose the busboy works an 8 hour shift that happens to begin as soon as the first customer of the day sits at the table.

   (a) What is the expected number of times that the busboy has to clear the table during his shift?

   (b) What is the probability that he only has to clear the table at most once?

6. (20 points) A certain math exam has a maximum score of 800 points. It is found that 5% of students get a score of 700 or better. 100 students in a class take the exam. Let X be the number of students in the class who get a score of at least 700.

   (a) Find $\mathrm{Var}(X)$.

   (b) What is the probability that fewer than 2 students get at least 700 on the exam?

7. (20 points) (a) Suppose a continuous random variable X has distribution function given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^3 - x^2 + x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$$

Find $\mathrm{E}(X)$.

(b) Determine whether or not the following function is a density function of a continuous random variable:

$$f(x) = \begin{cases} 3x^2 - 3x - \frac{1}{2} & \text{if } 0 \le x \le 2 \\ 0 & \text{else} \end{cases}$$

8. (20 points) Sometimes professors make typos on the board during lecture. Suppose the number of typos a professor makes during Friday lectures is Poisson distributed with mean 1/2 per Friday lecture, and the number of typos the professor makes during Monday and Wednesday lectures is Poisson distributed with mean 1/5 per Monday/Wednesday lecture. Suppose there are 20 total Monday/Wednesday lectures and 10 Friday lectures in a quarter, and the number of typos in different lectures are all independent.

   (a) What is the expected number of typos made during the whole quarter?

   (b) What is the probability that at least one typo is made during the quarter?

9. (20 points) Theoretically, I.Q. scores are normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 15$.

   (a) What percentage of people have an I.Q. between 70 and 145?

   (b) Suppose 35 randomly chosen people are in a room. What is the probability that all of them have an I.Q. greater than 130? (Assume everyone's I.Q.s are independent of each other.)

   (c) In the room with 35 people, what is the expected value of the number of people with an I.Q. greater than 130? (Again, assume I.Q.s are all independent.)

10. (20 points) 100 people with a certain disease enter an experimental drug trial. 50 patients are given a new medicine, and the other 50 patients are given a placebo. It is found that 75% of patients who take the medicine recover from the disease, and 20% of patients who take the placebo recover from the disease.

    (a) Alex is a participant in the trial. What is the probability that he will recover from the disease?

    (b) What is the probability that a patient took the medicine given that the patient recovered from the disease?

11. The following game is played at a carnival with a fair four-sided numbered die (i.e. $P(1) = P(2) = P(3) = P(4) = \frac{1}{4}$). You roll the four-sided die 192 times, and they count the number of times you roll a 4. If you roll a 4 between 50 and 60 times, you win. Otherwise, you lose. The game costs \$10 to play. If you win, you get your \$10 back, plus an additional \$10. If you lose, the carnival worker keeps your \$10.

    (a) Use the normal approximation with the continuity correction to estimate the probability that you win the game.

    (b) Is this game a good deal for you to play? If so, what is the expected amount of money that you will win? If not, what is the expected amount of money that you will lose?