Homework 1, due Friday, August 30

Please turn in well-written solutions for the following problems:

(1) Let A, B, C, and D be sets. Suppose that $A \setminus B \subseteq C \cap D$, and suppose that $x \in A$. Prove that if $x \notin D$, then $x \in B$.

(2) Use induction to prove that

$$\forall n \in \mathbb{N},\ 1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

(3) Use induction to prove that

$$\forall n \in \mathbb{N},\ 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

(4) Let f be a function with domain D, and suppose that $S \subseteq D$. We define $f(S) = \{f(x) \mid x \in S\}$.

   (a) Let A and B be subsets of D. Prove that $f(A \cap B) \subseteq f(A) \cap f(B)$.

   (b) Give an example of a function f and sets A and B in the domain of f such that $f(A) \cap f(B) \not\subseteq f(A \cap B)$.

(5) Suppose that $f : A \to B$ and $g : B \to C$, and let $h = g \circ f$.

   (a) Prove that if f is onto and g is onto, then h is onto.

   (b) Prove that if h is onto, then g is onto.

   (c) Suppose that h is 1-1. Is it true that g is 1-1? Prove or give a counterexample.

   (d) Suppose that h is 1-1. Is it true that f is 1-1? Prove or give a counterexample.

(6) (GRE Problem) Let S be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. Consider the two binary operations $+$ and $\circ$ on S defined as pointwise addition and composition of functions, as follows.

$$(f + g)(x) = f(x) + g(x)$$
$$(f \circ g)(x) = f(g(x))$$

Which of the following statements are true?

I. $\circ$ is commutative.
II. $+$ and $\circ$ satisfy the left distributive law $f \circ (g + h) = (f \circ g) + (f \circ h)$.
III. $+$ and $\circ$ satisfy the right distributive law $(g + h) \circ f = (g \circ f) + (h \circ f)$.

(A) None (B) II only (C) III only (D) II and III only (E) I, II, and III