Homework 3, due Wednesday, September 18

Please turn in well-written solutions for the following:

(1) Let A and B be non-empty bounded subsets of $\mathbb{R}$.

   (a) Prove that if $\sup(A) < \inf(B)$, then A and B are disjoint.

   (b) Prove that if $\inf(B) < \sup(A)$, then there exist $a \in A$ and $b \in B$ such that $b < a$.

(2) Recall that a number r is rational if there exist $a, b \in \mathbb{Z}$ such that $r = \frac{a}{b}$. We say $r \in \mathbb{Q}$.

   (a) Prove that if $r, q \in \mathbb{Q}$, then $r + q \in \mathbb{Q}$.

   (b) Prove that if $r \in \mathbb{Q}$ and $s \notin \mathbb{Q}$, then $r + s \notin \mathbb{Q}$. (That is, the sum of a rational and an irrational is irrational.)

(3) Prove that the irrational numbers are dense in $\mathbb{R}$. That is, prove that for any $x, y \in \mathbb{R}$ with $x < y$, there exists $s \in \mathbb{R}\backslash\mathbb{Q}$ such that $x < s < y$.
(Hint: You may use without proof the fact that the rational numbers are dense in $\mathbb{R}$, as well as the results of the previous problem.)

(4) Define $f : (0, 2) \to \mathbb{R}$ by $f(x) = \dfrac{x^2 - 4}{x - 2}$. Prove that $\lim_{x\to 2} f(x)$ exists.

(5) Define $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ by $f(x) = \dfrac{|x|}{x}$.

   (a) Use an ε argument to prove that $\lim_{x\to 0^+} f(x) = 1$.

   (b) Use an ε argument to prove that $\lim_{x\to 0^-} f(x) = -1$.

(6) (GRE Problem) For any nonempty sets A and B of real numbers, let $A \cdot B$ be the set defined by

$$A \cdot B = \{xy \ : \ x \in A \text{ and } y \in B\}.$$

If A and B are nonempty bounded sets of real numbers and if $\sup(A) > \sup(B)$, then $\sup(A \cdot B) =$

(A) $\sup(A)\sup(B)$
(B) $\sup(A)\inf(B)$
(C) $\max\{\sup(A)\sup(B), \inf(A)\inf(B)\}$
(D) $\max\{\sup(A)\sup(B), \sup(A)\inf(B)\}$
(E) $\max\{\sup(A)\sup(B), \inf(A)\sup(B), \inf(A)\inf(B)\}$