Homework 4, due Thursday, February 15

Please turn in well-written solutions for the following:

(1) Let $D \subset \mathbb{R}$ and suppose $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are both continuous. Define $h : D \to \mathbb{R}$ by $h(x) = \max\{f(x), g(x)\}$. Prove that h is continuous on D.

(2) Consider the function

$$f(x) = \begin{cases} |x| & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

(a) Prove that f is continuous at 0.

(b) Prove that if $a \neq 0$, then f is not continuous at a.

(3) Suppose $E \subset D \subset \mathbb{R}$, and suppose $x_0 \in E$. Let $f : D \to \mathbb{R}$ be a function, and define $g : E \to \mathbb{R}$ by $g(x) = f(x)$ for all $x \in E$.

(a) Prove that if f is continuous at x_0, then g is continuous at x_0.

(b) Give an example of sets E and D and functions f and g as above, such that g is continuous at x_0 but f is NOT continuous at x_0.

(4) We say that x is a *fixed point* of f if $f(x) = x$.

(a) Suppose that $f : [0, 1] \to [0, 1]$ is a continuous function. Prove that there exists $c \in [0, 1]$ such that $f(c) = c$. (That is, show that f has a fixed point.)

(b) Suppose that f is L-continuous on $\mathbb{R}$ for some $L \in [0, 1)$. Prove that f cannot have two different fixed points.

(5) Suppose $f_1 : \mathbb{R} \to \mathbb{R}$ is L_1-continuous for some $L_1 \geq 0$, and $f_2 : \mathbb{R} \to \mathbb{R}$ is L_2-continuous for some $L_2 \geq 0$. Prove that $f_2 \circ f_1$ is L-continuous with $L = L_1 L_2$.

(6) Let $I \subset \mathbb{R}$ be some interval, and suppose that $f : I \to \mathbb{R}$ is L-continuous for some $L \geq 0$. Prove that f is uniformly continuous on I.