Homework 5, due Thursday, March 8

Please turn in well-written solutions for the following:

(1) Prove that the function $f(x) = |x|$ is not differentiable at $x = 0$.

(2) If $f(x) = \cos(x)$, prove that $f'(x) = -\sin(x)$.
(Hint: You may use the fact that

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x-y}{2}\right)\sin\left(\frac{x+y}{2}\right)$$

without proof.)

(3) Suppose that f is a function that is n-times differentiable on (a, b), and define $F(x) = xf(x)$. Find a formula for the nth derivative $F^{(n)}(x)$, and use induction to prove that your formula is correct.

(4) Suppose that $f_1, f_2, \ldots, f_n$ are differentiable on (a, b), and suppose that for some fixed $x \in (a, b)$, we have that all of the values $f_1(x)$, $f_2(x)$, ..., $f_n(x)$ are non-zero. Consider the product $g = f_1 f_2 f_3 \cdots f_n$. Prove that

$$\frac{g'(x)}{g(x)} = \frac{f_1'(x)}{f_1(x)} + \cdots + \frac{f_n'(x)}{f_n(x)}.$$

(Hint: It may be useful to find and prove a formula for g'.)

(5) Use the mean value theorem to prove that $f(x) = \sin(x)$ is L-continuous on $\mathbb{R}$ for $L = 1$.

(6) Recall that a function $f : I \to \mathbb{R}$ is α-Hölder-continuous on I, if there exists $C \geq 0$ such that for every $x, y \in I$, we have $|f(x) - f(y)| \leq C\,|x - y|^{\alpha}$.

Suppose that $\alpha > 1$ and that f is differentiable on I. Prove that if f is α-Hölder-continuous on I, then f is a constant function on I.