Homework 5, due Friday, October 25

Please turn in well-written solutions for the following:

(1) Prove that the function $f(x) = |x|$ is not differentiable at $x = 0$.

(2) If $f(x) = \cos(x)$, prove that $f'(x) = -\sin(x)$.
(Hint: You may use the fact that

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x-y}{2}\right)\sin\left(\frac{x+y}{2}\right)$$

without proof.)

(3) Suppose that f is a function that is n-times differentiable on (a, b), and define $F(x) = xf(x)$. Find a formula for the nth derivative $F^{(n)}(x)$, and use induction to prove that your formula is correct.

(4) Use the mean value theorem to prove that $f(x) = \sin(x)$ is L-continuous on $\mathbb{R}$ for $L = 1$.

(5) Recall that a function $f : I \to \mathbb{R}$ is α-Hölder-continuous on I, if there exists $C \geq 0$ such that for every $x, y \in I$, we have $|f(x) - f(y)| \leq C\,|x-y|^{\alpha}$.

Suppose that $\alpha > 1$ and that f is differentiable on I. Prove that if f is α-Hölder-continuous on I, then f is a constant function on I.

(6) (GRE Problem) If f is a continuously differentiable real-valued function defined on the open interval $(-1, 4)$ such that $f(3) = 5$ and $f'(x) \geq -1$ for all x, what is the greatest possible value of $f(0)$?

(A) 3 (B) 4 (C) 5 (D) 8 (E) 11

(7) (GRE Problem) The function $f : \mathbb{R} \to \mathbb{R}$ is defined as follows.

$$f(x) = \begin{cases} 3x^2 & \text{if } x \in \mathbb{Q} \\ -5x^2 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Which of the following is true?
(A) f is discontinuous at all $x \in \mathbb{R}$.
(B) f is continuous only at $x = 0$ and differentiable only at $x = 0$.
(C) f is continuous only at $x = 0$ and nondifferentiable at all $x \in \mathbb{R}$.
(D) f is continuous at all $x \in \mathbb{Q}$ and nondifferentiable at all $x \in \mathbb{R}$.
(E) f is continuous at all $x \notin \mathbb{Q}$ and nondifferentiable at all $x \in \mathbb{R}$.