Homework 6, due Wednesday, November 6

Please turn in well-written solutions for the following:

(1) Use an ε-based argument to show that $f(x) = \dfrac{1}{x^2}$ is integrable on $[1, 2]$. (In other words, you may not simply cite the fact that f is monotone or continuous to conclude integrability.)

(2) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and $f(x) \geq 0$ for all $x \in [a, b]$. Prove that if $\displaystyle\int_a^b f(x)\, dx = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

(3) (a) Suppose that $f : [a, b] \to \mathbb{R}$ is bounded on all of $[a, b]$ and is continuous everywhere except at a single point, $x_0 \in [a, b]$. Prove that f is integrable on $[a, b]$.

(b) Now suppose that $f : [a, b] \to \mathbb{R}$ is bounded on $[a, b]$ and is continuous at all but a finite amount of points, $\{x_1, x_2, x_3, \ldots, x_n\} \subset [a, b]$. Prove that f is integrable on $[a, b]$.

(4) Suppose that $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are both bounded, integrable functions. Prove that the product $h = fg$ is integrable on $[a, b]$.

(5) For each $k \in \mathbb{Z}^+$, define $x_k = \dfrac{k}{k+1}$. Then for each k, $0 < x_k < 1$, and moreover, we have that

$$0 < x_1 < x_2 < x_3 < \cdots < x_k < x_{k+1} < \cdots < 1,$$

and $\sup\{x_k \mid k \in \mathbb{Z}^+\} = 1$. Define a function $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, x_1) \cup [x_2, x_3) \cup [x_4, x_5) \cup \cdots \\ 0 & \text{if } x = 1 \text{ or } x \in [x_1, x_2) \cup [x_3, x_4) \cup \cdots \end{cases}$$

Prove that f is integrable on $[0, 1]$.

(6) (GRE Problem) A real-valued function f defined on $\mathbb{R}$ has the following property.
For every positive number ϵ, there exists a positive number δ such that

$$|f(x) - f(1)| \geq \epsilon \text{ whenever } |x - 1| \geq \delta.$$

This property is <u>equivalent</u> to which of the following statements about f?

(A) f is continuous at $x = 1$.
(B) f is discontinuous at $x = 1$.
(C) f is unbounded.
(D) $\displaystyle\lim_{|x| \to \infty} |f(x)| = \infty$.
(E) $\displaystyle\int_0^\infty |f(x)|\, dx = \infty$.