Homework 7, due Tuesday, April 10

Please turn in well-written solutions for the following:

(1) Use the fundamental theorem of calculus and the rule for differentiating a product to prove the following "integration by parts" formula:
If u and v are differentiable functions on $[a, b]$ whose derivatives are continuous on $[a, b]$, then

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)|_a^b - \int_a^b u'(x)v(x)\,dx.$$

(2) Recall that we defined $L(x) = \displaystyle\int_1^x \frac{1}{t}\,dt$, we showed that L is a bijection from $(0, \infty)$ onto $\mathbb{R}$, and we defined E to be the inverse function of L. We showed that $E(0) = 1$, and that $E(x + y) = E(x)E(y)$ for all $x, y \in \mathbb{R}$. Then we used this to define the operation of raising a positive real number x to the power of any other real number s by

$$x^s = E(sL(x)).$$

(a) Under this definition and the properties of E listed above, show that for any s, $t \in \mathbb{R}$, we have that $x^{s+t} = x^s x^t$.

(b) We also showed that $E'(x) = E(x)$ for any real number x. Let s be an arbitrary nonzero real number, and define $f : [0, \infty) \to \mathbb{R}$ by $f(x) = x^s$. Use the above definition of x^s based on E, the differentiation property of E, and the chain rule to show that $f'(x) = sx^{s-1}$.

(3) Note that for any real numbers α and β, we have that

$$\int_a^b (\alpha f(x) + \beta g(x))^2\,dx \geq 0.$$

(a) Use this to prove that for any α, $\beta \in \mathbb{R}$ and for any integrable functions f and g on $[a, b]$, we have that

$$2\alpha\beta \int_a^b f(x)g(x)\,dx \leq \alpha^2 \int_a^b (f(x))^2\,dx + \beta^2 \int_a^b (g(x))^2\,dx.$$

(b) By picking the values of α and β in a clever way, use part (a) to show that

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \leq \left(\int_a^b (f(x))^2\,dx\right)\left(\int_a^b (g(x))^2\,dx\right).$$

This result is an integral form of the Cauchy-Schwarz Inequality.

(c) Use part (b) to show that

$$\left(\int_a^b (f(x) + g(x))^2\,dx\right)^{1/2} \leq \left(\int_a^b (f(x))^2\,dx\right)^{1/2} + \left(\int_a^b (g(x))^2\,dx\right)^{1/2}.$$

This result is a special case of Minkowski's Inequality.