

HOMEWORK 7, DUE FRIDAY, NOVEMBER 15

Please turn in well-written solutions for the following:

- (1) Use the fundamental theorem of calculus and the rule for differentiating a product to prove the following “integration by parts” formula:
If u and v are differentiable functions on $[a, b]$ whose derivatives are continuous on $[a, b]$, then

$$\int_a^b u(x)v'(x) dx = u(x)v(x)|_a^b - \int_a^b u'(x)v(x) dx.$$

- (2) Recall that we defined $L(x) = \int_1^x \frac{1}{t} dt$, we showed that L is a bijection from $(0, \infty)$ onto \mathbb{R} , and we defined E to be the inverse function of L . We showed that $E(0) = 1$, and that $E(x+y) = E(x)E(y)$ for all $x, y \in \mathbb{R}$. Then we used this to define the operation of raising a positive real number x to the power of any other real number s by

$$x^s = E(sL(x)).$$

- (a) Under this definition and the properties of E listed above, show that for any $s, t \in \mathbb{R}$, we have that $x^{s+t} = x^s x^t$.
- (b) We also showed that $E'(x) = E(x)$ for any real number x . Let s be an arbitrary nonzero real number, and define $f : [0, \infty) \rightarrow \mathbb{R}$ by $f(x) = x^s$. Use the above definition of x^s based on E , the differentiation property of E , and the chain rule to show that $f'(x) = sx^{s-1}$.

- (3) Note that for any real numbers α and β , we have that

$$\int_a^b (\alpha f(x) + \beta g(x))^2 dx \geq 0.$$

- (a) Use this to prove that for any $\alpha, \beta \in \mathbb{R}$ and for any integrable functions f and g on $[a, b]$, we have that

$$2\alpha\beta \int_a^b f(x)g(x) dx \leq \alpha^2 \int_a^b (f(x))^2 dx + \beta^2 \int_a^b (g(x))^2 dx.$$

- (b) By picking the values of α and β in a clever way, use part (a) to show that

$$\left(\int_a^b f(x)g(x) dx \right)^2 \leq \left(\int_a^b (f(x))^2 dx \right) \left(\int_a^b (g(x))^2 dx \right).$$

This result is an integral form of the Cauchy-Schwarz Inequality.

- (c) Use part (b) to show that

$$\left(\int_a^b (f(x) + g(x))^2 dx \right)^{1/2} \leq \left(\int_a^b (f(x))^2 dx \right)^{1/2} + \left(\int_a^b (g(x))^2 dx \right)^{1/2}.$$

This result is a special case of Minkowski's Inequality.