HOMEWORK 8, DUE THURSDAY, APRIL 26

Please turn in well-written solutions for the following:

(1) (a) Give an example of sequences (a_n) and (b_n) that do not converge but such that $(a_n + b_n)_{n\in\mathbb{Z}^+}$ does converge.

(b) Give an example of sequences (a_n) and (b_n) that do not BOTH converge (one of them may) but such that $(a_n b_n)_{n\in\mathbb{Z}^+}$ does converge.

(c) Prove that if (a_n) converges to $A \neq 0$ and (b_n) is a sequence such that $(a_n b_n)$ converges, then (b_n) must also converge.

(2) (a) Let $(b_n)_{n\in\mathbb{Z}^+}$ be a sequence such that (b_n) converges to B, and $B \neq 0$. Prove that there exists $M > 0$ and $N \in \mathbb{Z}^+$ such that if $n \geq N$, then $|b_n| \geq M$.

(b) Suppose (a_n) converges to A and (b_n) converges to B, with $B \neq 0$ and $b_n \neq 0$ for all n. Prove that $\left(\dfrac{a_n}{b_n}\right)_{n=1}^{\infty}$ converges to $\dfrac{A}{B}$.

(3) Let (s_n) be a convergent sequence with the property that exactly one trillion terms in the sequence have absolute value strictly greater than 1 (not necessarily the first one trillion terms, they are scattered throughout the sequence). Let $L = \lim_{n\to\infty} s_n$. Prove that $|L| \leq 1$.

(4) Let S be a nonempty set of real numbers, and suppose that $\sup(S) = M$ exists. Prove that there exists a sequence $(s_n)_{n\in\mathbb{Z}^+}$ such that $s_n \in S$ for each $n \in \mathbb{Z}^+$ and such that $M = \lim_{n\to\infty} s_n$. (That is, M is an *accumulation point* of S.)

(5) The Fibonacci sequence is defined by $f_0 = 0$, $f_1 = 1$, and for any $n \geq 2$, we define f_n recursively by $f_n = f_{n-1} + f_{n-2}$. We use this to define a new sequence by $r_n = \dfrac{f_{n+1}}{f_n}$.

(a) Use some simple algebra to show that $r_{n+1} = 1 + \dfrac{1}{r_n}$ for every n.

(b) Assume that (r_n) is a convergent sequence. (This is true, but we may not have time to get through all the steps needed to show it.) In this case, use (a) to show that $\lim_{n\to\infty} r_n = \dfrac{\sqrt{5}+1}{2}$, the golden ratio.