Homework 1, due Friday, January 24

Please turn in well-written solutions for the following problems:

(1) (1.1.13 in Tao) Let X be any set, $x \in X$, and $(x_n)_{n=1}^{\infty}$ a sequence in X. Prove that (x_n) converges to x in the discrete metric d_{disc} if and only if there exists N such that $x_n = x$ for every $n \geq N$.

(2) (1.1.16 in Tao) Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be two sequences in some metric space (X, d), such that $x_n \to x$ and $y_n \to y$ for some points $x, y \in X$. Prove that $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$. (Hint: Use the triangle inequality multiple times.)

(3) Let X be the set of all continuous real-valued functions with domain $[0, 1]$. Define a function $d : X \times X \to [0, \infty)$ by $d(f, g) = \int_0^1 (f(x) - g(x))^2 dx$, for any f and g in X. Prove that (X, d) is NOT a metric space, because the triangle inequality is not satisfied. (Hint: This means you have to find counter-examples of functions that make the triangle inequality fail. Consider constant functions.)

(4) Consider $\mathbb{R}^2$ with the metrics d_{l^2}, d_{l^1}, d_{l^∞}, and d_{disc}. In each of these metrics, sketch $B((0, 0), 1)$, the ball of radius 1 centered at the origin. That is, I want you to:
   (i) Sketch $B_{(\mathbb{R}^2, d_{l^2})}((0, 0), 1)$.
   (ii) Sketch $B_{(\mathbb{R}^2, d_{l^1})}((0, 0), 1)$.
   (iii) Sketch $B_{(\mathbb{R}^2, d_{l^\infty})}((0, 0), 1)$.
   (iv) Sketch $B_{(\mathbb{R}^2, d_{\text{disc}})}((0, 0), 1)$.

(5) Let (X, d) be a metric space, and let $E, F \subset X$.
   (a) Prove that $\text{int}(E) \cup \text{int}(F) \subset \text{int}(E \cup F)$.
   (b) Give an example of a metric space X with subsets E and F such that $\text{int}(E) \cup \text{int}(F) \neq \text{int}(E \cup F)$. (That is, $\text{int}(E \cup F)$ contains points that are in neither $\text{int}(E)$ nor $\text{int}(F)$. Don't overthink it! This can be done even with a metric space as simple as $X = \mathbb{R}$.)

(6) Let (X, d) be a metric space, and let $E \subset X$. Prove that $x \in \partial E$ if and only if for every $r > 0$, $B(x, r) \cap E \neq \emptyset$ and $B(x, r) \cap E^c \neq \emptyset$.

In addition, I suggest that you study these problems from Tao:

- Section 1.1, problems 1.1.4, 1.1.5, 1.1.6, 1.1.12
- Section 1.2, problems 1.2.1, 1.2.4