Homework 2, due Friday, August 31

Please turn in well-written solutions for the following problems:

(1) Let (X, d) be any metric space.
   (a) Prove that $B_{(X,d)}(x_0, r)$ is open for any $x_0 \in X$ and any $r > 0$.

   (b) Prove that $\{x \in X \,:\, d(x, x_0) \leq r\}$ is closed for any $x_0 \in X$, $r > 0$.

(2) Let (X, d) be a metric space, and let $E, F \subset X$. Recall the notation that $A \backslash B$ is the set of all elements in A that are not elements in B. Using the definitions and theorems given in class, prove the following:
   (a) $\overline{E} = E \cup \partial E$
   (b) $\mathrm{int}(E) = (\overline{E^c})^c$

(3) (1.4.5 in Tao) Let (X, d) be a metric space, $E \subseteq X$. Recall that we defined x to be a *limit point* or *adherent point* of a set E if there exists a sequence $(x_n)_{n=1}^{\infty}$ in E such that $x_n \to x$. Recall that we defined x to be a *cluster point* of a sequence $(x_n)_{n=0}^{\infty}$ if for all $\varepsilon > 0$ and for all $N \geq 1$, there exists $n \geq N$ such that $d(x_n, x) < \varepsilon$.

   (a) Suppose that $(x_n)_{n=1}^{\infty}$ is a sequence in a metric space (X, d). Prove that if x is a cluster point of the sequence, then x is an adherent point of the set $\{x_n \,:\, n \geq 1\}$.

   (b) Prove that the converse is false. That is, prove that there exists a sequence $(x_n)_{n=1}^{\infty}$ in some metric space (X, d) such that x is an adherent point of the set $\{x_n \,:\, n \geq 1\}$, but such that x is not a cluster point of the sequence.

(4) (1.4.6 in Tao) Let (X, d) be a metric space, and $(x_n)_{n=1}^{\infty}$ be a sequence in X. Prove that if L_1 and L_2 are both cluster points of (x_n) and $L_1 \neq L_2$, then (x_n) is not Cauchy.

In addition, I suggest that you study these problems from Tao:

- Section 1.4, problems 1.4.3, 1.4.2, 1.4.8