

HOMWORK 3, DUE FRIDAY, SEPTEMBER 7

Please turn in well-written solutions for the following problems:

- (1) (1.5.9 in Tao) Show that a metric space  $(X, d)$  is compact if and only if every sequence in  $X$  has at least one cluster point.
- (2) (1.5.12 in Tao) Let  $(X, d_{\text{disc}})$  be a metric space with the discrete metric.
  - (a) Prove that  $(X, d_{\text{disc}})$  is always complete.
  - (b) Under what conditions on the set  $X$  do we have that  $(X, d_{\text{disc}})$  is compact? When is  $(X, d_{\text{disc}})$  not compact?
- (3) (1.5.14 in Tao) Let  $(X, d)$  be a metric space, let  $E \subseteq X$  be compact and non-empty, and let  $x_0 \in X$ . Show that there exists a point  $x \in E$  such that

$$d(x_0, x) = \inf\{d(x_0, y) : y \in E\},$$

i.e.,  $x$  is the closest point in  $E$  to  $x_0$ . (Hint: Let  $R = \inf\{d(x_0, y) : y \in E\}$ , and construct a sequence  $(x_n)$  in  $E$  with  $d(x_0, x_n) \leq R + \frac{1}{n}$ . Then use compactness.)

- (4) (1.5.15 in Tao plus more) Let  $(X, d)$  be a compact metric space. Suppose that  $(K_\alpha)_{\alpha \in I}$  is a collection of closed sets in  $X$  such that any finite sub-collection of these sets has non-empty intersection. That is, for any finite set  $F \subseteq I$ , we have that  $\bigcap_{\alpha \in F} K_\alpha \neq \emptyset$ . (This property is called the *finite intersection property*.)
  - (a) Prove that if  $(K_\alpha)_{\alpha \in I}$  has the finite intersection property, then the grand intersection  $\bigcap_{\alpha \in I} K_\alpha$  is non-empty.
  - (b) Show by counterexample that part (a) is false if  $X$  is not compact.
  - (c) Prove, however, that if every collection of closed subsets of  $X$  that has the finite intersection property also has a non-empty grand intersection, then  $X$  must be compact. (Hint: Given an open covering  $(V_\alpha)_{\alpha \in I}$ , consider the collection of closed sets given by  $(V_\alpha^c)_{\alpha \in I}$ .)

In addition, I suggest that you study these problems from Tao:

- Section 1.5, problems 1.5.2, 1.5.8, 1.5.10 (not easy!), 1.5.13