Homework 4, due Wednesday, September 19

Please turn in well-written solutions for the following problems:

(1) (2.1.7 in Tao) Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f : X \to Y$. Suppose that the image $f(X)$ of X is contained in E, for some $E \subseteq Y$. Let $g : X \to E$ be the function which is the same as f, but with the codomain restricted from Y to E, thus $g(x) = f(x)$ for all $x \in X$. Consider $(E, d_Y\big|_{E\times E})$, the induced space. Show that $f : X \to Y$ is continuous if and only if $g : X \to E$ is continuous.

(2) Let $(X, d_X), (Y, d_Y)$ be metric spaces. We define a function $f : X \to Y$ to be *open* if for each open set $V \subset X$, the image $f(V)$ is open in Y.
  (a) If f is open, is it continuous? Prove or find a counterexample.
  (b) If f is continuous, is it open? Prove or find a counterexample.

(3) (2.3.2 in Tao) Let (X, d) be a compact metric space, and let $f : X \to \mathbb{R}$ be a continuous function. Let $M = \sup\{f(x) : x \in X\}$.
  (a) Prove that f is bounded. (This ensures that $M < \infty$.)
  (b) Prove that there exists $x_{max} \in X$ such that $f(x_{max}) = M$. (Thus, f attains its maximum.)
  (c) Find a counterexample to prove that if X is not assumed to be compact, (b) does not necessarily hold. (Hint: A MATH 452 example will suffice!)

(4) (2.4.8 in Tao plus more) Let (X, d) be a metric space, and let $E \subseteq X$.
  (a) Show that if E is connected, then $\overline{E}$ is connected.
  (b) Is the converse true? That is, if E is disconnected, does it follow that $\overline{E}$ is disconnected?
  (c) You saw in Theorem 2.4.6 that if $f : X \to Y$ is a continuous function and $E \subseteq Y$ is connected, then $f(E)$ is connected. That is, f takes connected sets to connected sets. Is it also true that f takes disconnected sets to disconnected sets? Prove or find a counterexample.

In addition, I suggest that you study these problems from Tao:

- Section 2.1, problems 2.1.4, 2.1.5
- Section 2.2, problems 2.2.3, 2.2.10
- Section 2.3, problems 2.3.3, 2.3.4
- Section 2.4, problems 2.4.1, 2.4.2, 2.4.6, 2.4.7