Homework 5, due Friday, October 5

Please turn in well-written solutions for the following problems:

(1) (3.2.2 (a) and (b) in Tao)
  (a) Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and let $f : X \to Y$ be another function. Show that if $f_n \to f$ uniformly, then $f_n \to f$ pointwise.
  (b) For each positive integer n, define $f_n : (0, 1) \to \mathbb{R}$ by $f_n(x) = x^n$. Prove that $(f_n)_{n=1}^{\infty}$ does not converge uniformly to any function on $(0, 1)$.

(2) (3.2.4 in Tao) Let $(X, d_X), (Y, d_Y)$ be metric spaces, and let $f_n : X \to Y$ be a bounded function for each n, and suppose that $f_n \to f$ uniformly, for some bounded function $f : x \to Y$. Prove that the sequence is *uniformly bounded*. That is, prove that there exists a ball $B_{(Y,d_Y)}(y_0, R)$ in Y such that for all $x \in X$ and for all positive integers n, we have that $f_n(x) \in B(y_0, R)$. (Notably, y_0 and R do not depend on x or on n.)

(3) (3.3.8 in Tao) Let (X, d_X) be a metric space, and consider two sequences of functions $(f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty} \subset B(X \to \mathbb{R})$. Suppose further that both sequences are uniformly bounded, as defined in the previous problem. (So since the codomain is now the real numbers, this means that there exists an $M > 0$ such that $|f_n(x)| \leq M$ and $|g_n(x)| \leq M$ for all $x \in X$ and every positive integer n.) If $f_n \to f$ uniformly and $g_n \to g$ uniformly, prove that the sequence of products $(f_n g_n)_{n=1}^{\infty}$ converges uniformly to fg.

(4) (3.3.7 in Tao) Find an example of a sequence of bounded functions $(f_n)_{n=1}^{\infty}$ from $\mathbb{R}$ to $\mathbb{R}$ such that $f_n \to f$ pointwise on $\mathbb{R}$, but f is not bounded.

(5) (3.4.4 in Tao, modified) Let (X, d_X) and (Y, d_Y) be metric spaces. Define $Y^X = \{f : X \to Y\}$, the space of all functions from X to Y. It turns out that there isn't a good metric to put on Y^X to make it into an interesting metric space, but we still have a way of defining open sets and convergence in Y^X without a metric.
To this end, if $x_0 \in X$ and $V \subseteq Y$ is an open set, we define
$V^{(x_0)} = \{f \in Y^X : f(x_0) \in V\}$. Then if $E \subseteq Y^X$, we say that E is *open in* Y^X if for every $f \in E$, there exists a finite number of points $x_1, x_2, \ldots, x_n \in X$ and open sets $V_1, V_2, \ldots V_n \subseteq Y$ such that

$$f \in V_1^{(x_1)} \cap V_2^{(x_2)} \cap \ldots \cap V_n^{(x_n)} \text{ and } V_1^{(x_1)} \cap V_2^{(x_2)} \cap \ldots \cap V_n^{(x_n)} \subseteq E.$$

(Note: For us, the major takeaway from this is that the sets $V_i^{(x_i)}$ are, in fact, open sets, so this property is roughly analogous to the metric space idea that a set is open if every point in the set has an open ball around it completely contained in the set.)
For each positive integer n, let $f_n \in Y^X$, and let $f \in Y^X$. Prove that $f_n \to f$ pointwise on X if and only if for every open set $E \subseteq Y^X$ with $f \in E$, there exists N such that whenever $n \geq N$, we have $f_n \in E$.
(Hint: Really work on this one! Despite the fact that we're defining some new things here, when you break down all the definitions, it's not too bad!)

In addition, I suggest that you study these problems from Tao:

- Section 3.1, problem 3.1.5
- Section 3.2, problem 3.2.3
- Section 3.3, problems 3.3.4, 3.3.5
- Section 3.4, problem 3.4.3