HOMEWORK 6, DUE FRIDAY, OCTOBER 19

Please turn in well-written solutions for the following problems:

(1) For each positive integer n, let $f_n : [-\pi, \pi] \to \mathbb{R}$ by $f_n(x) = \cos(x + n)$. Show that $(f_n)_{n=1}^{\infty}$ has a subsequence that converges uniformly to a continuous function on $[-\pi, \pi]$.

(2) For a metric space (X, d_X), we say that a continuous function $f : X \to \mathbb{R}$ *vanishes at infinity* if for every $\varepsilon > 0$, there exists a compact set $K \subseteq X$ such that for all $x \in K^c$, we have that $|f(x)| < \epsilon$. We define the set

$$C_0(X, \mathbb{R}) = \{f : X \to \mathbb{R} |\ f \text{ is continuous and vanishes at infinity}\}.$$

(a) Show that $C_0(X, \mathbb{R}) \subseteq B(X, \mathbb{R})$.
(b) We showed in class that $C(X, Y)$ is always closed in $(B(X, Y), d_\infty)$. Is $C_0(X, \mathbb{R})$ closed in $(B(X, \mathbb{R}), d_\infty)$? Prove or give a counterexample.

(3) In class, we used the word equicontinuity of a collection F of functions from X to Y to refer to *uniform equicontinuity*, which says that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $f \in F$ and for all $x_1, x_2 \in X$, whenever $d(x_1, x_2) < \delta$ we have that $d(f(x_1), f(x_2)) < \varepsilon$.
However, for a point $x_0 \in X$, we also give the following definition. We say that F is *pointwise equicontinuous at* x_0 if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $f \in F$ and for all $x \in X$, whenever $d(x_0, x) < \delta$ we have that $d(f(x_0), f(x)) < \varepsilon$. Then we say that F is *pointwise equicontinuous on* X if F is pointwise equicontinuous at x_0 for every $x_0 \in X$.

(a) Suppose that X is a compact metric space and F is a collection of functions from X to Y. Prove that if F is pointwise equicontinuous on X, then F is uniformly equicontinuous.
(b) Is the converse of (a) true? That is, if every pointwise equicontinuous sequence of functions is automatically uniformly equicontinuous, does that imply that X is compact? Prove or give a counterexample.

(4) Let (X, d_X) be a compact metric space. Recall that a function $f : X \to X$ that preserves distances is called an *isometry*; that is, f is an isometry if for every $x_1, x_2 \in X$, we have that $d(x_1, x_2) = d(f(x_1), f(x_2))$. Let $\text{Iso}(X) = \{f : X \to X |\ f \text{ is an isometry}\}$. (Note: This is not any sort of standard notation for the set of isometries; I'm just making it up now.)

(a) Prove that $\text{Iso}(X) \subseteq B(X, X)$. (In fact, $\text{Iso}(X) \subseteq C(X, X)$.)
(b) Prove that $\text{Iso}(X)$ is a compact subset of $(B(X, X), d_\infty)$.