# Homework 6, due Friday, March 13

Please turn in well-written solutions for the following problems:

(1) For each positive integer $n$, let $f_n : [-\pi, \pi] \to \mathbb{R}$ by $f_n(x) = \cos(x+n)$. Show that $(f_n)_{n=1}^{\infty}$ has a subsequence that converges uniformly to a continuous function on $[-\pi, \pi]$.

(2) In class, we used the word equicontinuity of a collection $F$ of functions from $X$ to $Y$ to refer to *uniform equicontinuity*, which says that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $f \in F$ and for all $x_1, x_2 \in X$, whenever $d(x_1, x_2) < \delta$ we have that $d(f(x_1), f(x_2)) < \varepsilon$.
However, for a point $x_0 \in X$, we also give the following definition. We say that $F$ is *pointwise equicontinuous at* $x_0$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $f \in F$ and for all $x \in X$, whenever $d(x_0, x) < \delta$ we have that $d(f(x_0), f(x)) < \varepsilon$. Then we say that $F$ is *pointwise equicontinuous on* $X$ if $F$ is pointwise equicontinuous at $x_0$ for every $x_0 \in X$.
   (a) Suppose that $X$ is a compact metric space and $F$ is a collection of functions from $X$ to $Y$. Prove that if $F$ is pointwise equicontinuous on $X$, then $F$ is uniformly equicontinuous.
   (b) The statement in (a) is false if $X$ is not assumed to be compact. Show this by finding a counterexample on $\mathbb{R}$. That is, find a collection of functions $F$ from $\mathbb{R}$ to $\mathbb{R}$ that is pointwise equicontinuous on $\mathbb{R}$ but is not uniformly equicontinuous. (Hint: Your collection $F$ doesn't need to have many functions in it.)
   (c) Is the converse of (a) true? That is, if every pointwise equicontinuous sequence of functions is automatically uniformly equicontinuous, does that imply that $X$ is compact? Prove or give a counterexample.

(3) Recall that we defined $\mathrm{Lip}_K(X \to \mathbb{R})$ to be the set of all functions $f$ in $B(X \to \mathbb{R})$ such that $|f(x_1) - f(x_2)| \leq K\, d_X(x_1, x_2)$ for all $x_1, x_2 \in X$, where $K > 0$. Suppose that $X$ is compact.
   (a) Prove that $\mathrm{Lip}_K(X \to \mathbb{R})$ is equicontinuous.
   (b) Prove that $\mathrm{Lip}_K(X \to \mathbb{R})$ is not uniformly bounded. In particular, find a sequence $(f_n)_{n=1}^{\infty}$ of functions such that $f_n \in \mathrm{Lip}_K$ for every $n$, but that $\lim\limits_{n\to\infty} \sup\{f_n(x) \,:\, x \in X\} = \infty$.

(4) Let $(X, d_X)$ be a compact metric space. We say that $f : X \to X$ is an *isometry* if for every $x_1, x_2 \in X$, we have that $d(x_1, x_2) = d(f(x_1), f(x_2))$. Let $\mathrm{Iso}(X) = \{f : X \to X \mid f \text{ is an isometry}\}$. (Note: This is not any sort of standard notation for the set of isometries; I'm just making it up now.)
   (a) Prove that $\mathrm{Iso}(X) \subseteq B(X \to X)$. (In fact, $\mathrm{Iso}(X) \subseteq C(X \to X)$.)
   (b) Prove that $\mathrm{Iso}(X)$ is a compact subset of $(B(X \to X), d_\infty)$.