Homework 7, due Wednesday, April 15

Please turn in well-written solutions for the following problems:

(1) Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions, $f_n : [0,1] \to \mathbb{R}$ such that for each $x \in [0,1]$, $f_n(x) \to 0$ as $n \to \infty$. Suppose further that there exists a constant K such that for every n,

$$\left| \int_0^1 f_n(x)\, dx \right| \leq K.$$

Is it true that $\int_0^1 f_n(x)\, dx \to 0$ as $n \to \infty$? Prove or give a counterexample.

(2) For a metric space (X, d_X), we say that a bounded continuous function $f : X \to \mathbb{R}$ *vanishes at infinity* if for every $\varepsilon > 0$, there exists a compact set $K \subseteq X$ such that for all $x \in K^c$, we have that $|f(x)| < \epsilon$. We define the set

$$C_0(X, \mathbb{R}) = \{f \in B(X, \mathbb{R}) \mid f \text{ is continuous and vanishes at infinity}\}.$$

(It turns out that $C_0(X, \mathbb{R})$ is closed in $(B(X, \mathbb{R}), d_\infty)$.) Now, we make another definition. We say that a function $f : X \to \mathbb{R}$ is *compactly supported* if there exists a compact set $K \subseteq X$ such that for all $x \in K^c$, $f(x) = 0$. We define the set

$$C_c(X, \mathbb{R}) = \{f \in B(X, \mathbb{R}) \mid f \text{ is continuous and compactly supported}\}.$$

(a) Show that $C_c(X, \mathbb{R}) \subseteq C_0(X, \mathbb{R})$.

(b) Show that, in fact, $\overline{C_c(X, \mathbb{R})} = C_0(X, \mathbb{R})$, where the closure is taken with respect to the d_∞ metric.

(c) In light of part (b), is $C_c(X, \mathbb{R})$ a closed set in $(B(X, \mathbb{R}), d_\infty)$? Prove or give a counterexample.

(3) Define $f_n : [1,2] \to \mathbb{R}$ by $f_n(x) = \dfrac{x}{(1+x)^{n+1}}$.

Prove that $\sum_{n=0}^{\infty} f_n$ is uniformly convergent on $[1,2]$.

(4) (3.8.8 in Tao) Let $f : [0,1] \to \mathbb{R}$ be a continuous function, and suppose that for every non-negative integer n, we have that $\int_0^1 f(x) x^n dx = 0$. Prove that $f(x) = 0$ for all $x \in [0,1]$. (Hint: first show that $\int_0^1 f(x) P(x) dx = 0$ for every polynomial P, then use the Weierstrass approximation theorem to show that $\int_0^1 (f(x))^2 dx = 0$.)

In addition, I suggest that you study these problems from Tao:

- Section 3.6, problem 3.6.1
- Section 3.7, problem 3.7.2