Homework 8, due Friday, November 16

Please turn in well-written solutions for the following problems:

(1) (4.2.2 in Tao) Show that the function $f(x) = \frac{1}{1-x}$ is real analytic on all of $\mathbb{R}\backslash\{1\}$.

(2) (4.5.7 in Tao) Let $f : \mathbb{R} \to (0, \infty)$ be a real analytic function such that $f'(x) = f(x)$ for all $x \in \mathbb{R}$. Prove that $f(x) = Ce^x$ for some $C > 0$.

(3) (4.5.8/4.5.9 in Tao)
   (a) Let $m > 0$ be an integer. Show that
   $$\lim_{x\to+\infty} \frac{e^x}{x^m} = +\infty.$$
   (Hint: Consider the ratio between $e^{x+1}/(x + 1)^m$ and e^x/x^m as x increases.)

   (b) Let $P(x)$ be any polynomial, and let $c > 0$ be arbitrary. Show that there exists $N > 0$ such that whenever $x > N$, we have $e^{cx} > |P(x)|$. That is, every exponential function always eventually beats any polynomial function.

(4) (4.6.15 in Tao) In the real numbers, we have positive numbers, negative numbers, and zero. The purpose of this exercise is to explain why, unfortunately, the complex numbers do not have this property. The nice thing about the real numbers is that we can define a positive number and a negative number in a manner that obeys the following axioms:
   - (Trichotomy) For any number x, exactly one of the following is true: x is positive, x is negative, x is zero.
   - (Negation) If x is positive, then $-x$ is negative. If x is negative, then $-x$ is positive.
   - (Additivity) If x and y are positive, then $x + y$ is also positive.
   - (Multiplicativity) If x and y are positive, then xy is also positive.

   We will show that in the complex numbers, we cannot define positive and negative numbers in a manner that satisfy these axioms. To that end, assume for sake of contradiction that we DO have a definition of positive and negative complex numbers satisfying these axioms.
   (a) Use the axioms to show that 1 must be positive. (Note: This is the most difficult part!)
   (b) Use (a) to conclude that -1 is negative.
   (c) By trichotomy, i must be positive, negative, or zero. It is not zero, so it must be positive or negative. Use (b) and multiplicativity to show that i cannot be positive.
   (d) By trichotomy and (c), i must be negative. Show that this case also leads to a contradiction.

   (Then since i cannot be positive, negative, or zero, we get that the axioms cannot be satisfied.)

In addition, I suggest that you study these problems from Tao:

- Section 4.5, problems 4.5.2, 4.5.4, 4.5.6
- If you are very unfamiliar with the complex numbers, all of 4.6.1-4.6.14 are good exercises for practice.