Final Exam Material and Expectations

For the final exam, you should be able to do the following things:

Chapter 1.

- Add, subtract, multiply, and divide complex numbers
- Find complex conjugate, modulus, and argument of a complex number
- Convert between $x + iy$ form and polar form $re^{i\theta}$
- Multiply, divide, and take powers and roots of complex numbers in polar form
- Solve basic equations in $\mathbb{C}$
- Sketch regions in the complex plane
- Determine whether regions are open, closed, connected, bounded, etc.

Chapter 3.

- Compute values of functions evaluated at complex numbers, including polynomials, rational functions, complex exponential, and branches of complex logarithms and radicals
- Write general form of a multivalued function such as logarithms or radicals in terms of an arbitrary integer n
- Determine largest domain of definition of a function

Chapter 2.

- Compute limits in $\mathbb{C}$, and determine when limits do not exist
- Compute limits involving the point at ∞
- Use derivative formulas for complex functions that mirror the formulas of the real analogues
- Use the Cauchy-Riemann equations to determine whether a complex function given in terms of real and imaginary parts is analytic at a point
- Use Cauchy-Riemann equations to find harmonic conjugates. That is, given $u(x, y)$ harmonic, find $v(x, y)$ harmonic such that $f(x, y) = u(x, y)+iv(x, y)$ is analytic

Chapter 4.

- Compute integrals of functions from $\mathbb{R}$ to $\mathbb{C}$
- Write typical parametrizations of simple contours such as line segments or arcs of a circle
- Use parametrization to write an integral of a complex function over a contour in $\mathbb{C}$ as an integral over a real interval:

$$\int_C f(z)\, dz = \int_a^b f(z(t))z'(t)\, dt$$

- Use the Cauchy-Goursat Theorem and related results:
  - If f has an antiderivative F on a domain containing the contour C from z_1 to z_2, then $\int_C f(z)\,dz = F(z_2) - F(z_1)$
  - f is analytic everywhere in a simply connected domain D if and only if f has an antiderivative on D
  - If f is analytic on and inside a closed simple contour C, then $\int_C f(z)dz = 0$
- Use the Cauchy Integral Formula: If f is analytic on and inside a simple closed positive contour C around a point z_0, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz \text{ and for any } n \in \mathbb{N}, f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz$$

Chapter 5.

- Determine convergence/limit of sequences or series of complex numbers by comparing to the sequences or series of real and imaginary parts
- Use the geometric series and the Maclaurin series of e^z, $\sin(z)$, and $\cos(z)$:
  - $\dfrac{1}{1-z} = \displaystyle\sum_{n=0}^{\infty} z^n$ for $|z| < 1$
  - $e^z = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$ for any $z \in \mathbb{C}$
  - $\sin(z) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$ for any $z \in \mathbb{C}$
  - $\cos(z) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$ for any $z \in \mathbb{C}$
- Use the above Maclaurin series to write Laurent series of rational functions and functions involving exponentials and trig functions

Chapter 6.

- Find singularities of a function and determine whether they are removable, essential, poles, or non-isolated
- Compute the residue of a function at an essential singularity using Laurent series
- Compute the residue of a function at a pole of higher order by Laurent series or by writing $f(z) = \phi(z)(z - z_0)^{-m}$ and calculating

$$\text{Res}(f, z_0) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

- Compute the residue of a function at a simple pole by either of the above techniques, or using

$$\text{Res}\left(\frac{p(z)}{q(z)}, z_0\right) = \frac{p(z_0)}{q'(z_0)}$$

- Compute integrals using the Cauchy Residue Theorem: If f has only finitely many singularities interior to a positive simple closed contour C, then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, z_k)$$

Chapter 7.

- Use Cauchy Residue Theorem and semicircular contours to compute improper integrals of real-valued functions over $\mathbb{R}$
- Use Cauchy Residue Theorem, semicircular contours, and e^{ix} to compute improper integrals from Fourier analysis
- Use Cauchy Residue Theorem, indented semicircular contours, and complex functions with branch cuts to compute improper integrals of real-valued functions with logarithms or radicals
- Use Cauchy Residue Theorem and substitutions for $\sin(\theta)$ and $\cos(\theta)$ to compute integrals of functions involving sin and cos over $[0, 2\pi]$

Chapter 8.

- Given two sets of three distinct points, find the unique Mobius transformation that maps the first set of points to the second set of points
- Given a function and a set S, determine the image set $f(S)$