Homework 3, due Friday, February 26

Please turn in solutions for the following problems:

(1) Find each limit or explain why it does not exist:

(a) $\displaystyle\lim_{z\to -2i} \frac{z^3-8i}{z+2i}$

(b) $\displaystyle\lim_{z\to 8+i} \frac{1}{1-\mathrm{Im}(z)}$

(c) $\displaystyle\lim_{z\to\infty} \frac{4z^6-7z^3}{(z^2-4)^3}$

(d) $\displaystyle\lim_{z\to\infty} \frac{|z|}{z}$

(2) Use the rules for differentiation to find the derivative of each function.

(a) $f(z) = e^{z^3-z}$

(b) $f(z) = \cos^3(z^2)$

(c) $f(z) = \dfrac{z+1}{z+i}$, where $z \neq i$

(d) $f(z) = (\mathrm{Log}(z))^3$, where z is not on the negative real axis

(3) Let $g(z) = \overline{z}$. Write in the form $g(x+iy) = u(x+iy)+iv(x+iy)$. Check the Cauchy-Riemann equations to determine if this function is differentiable.

(4) Recall that we defined $\tan^{-1}(z) = \dfrac{i}{2}\log\left(\dfrac{i+z}{i-z}\right)$. Let $f(z) = \tan^{-1}(z)$, where we use the principal branch Log of the logarithm to define this as a single-valued function. Write in the form $f(x+iy) = u(x+iy)+iv(x+iy)$. (That is, find u and v.) Then, check the Cauchy-Riemann equations to determine if this function is differentiable. If so, find the derivative, $f'(z)$.

(5) Suppose that f is an entire function such that $f(z) = u(z) + iv(z)$, where $u(x+iy) = 2x^2+2x+1-2y^2$. Determine what v must be.

In addition, I suggest that you work these problems from the Brown/Churchill textbook (but do not turn in):

- Pages 55-56, problems 3, 10
- Page 62, problem 1
- Page 71, problems 2, 4
- Pages 77, problems 1, 2
- Page 81, problem 1

Optional Proofs (not for credit):

- Define $u(x, y) = \sqrt{|xy|}$, and $v(x, y) = 0$ for all $x, y \in \mathbb{R}$. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(x+iy) = \sqrt{|xy|}$. Show that f satisfies the Cauchy-Riemann equations at 0, but is not differentiable at 0.

- Let f be analytic on a domain D and suppose that $f'(z) = 0$ for all $z \in D$. Show that f is constant on D.

- Suppose f is analytic on a domain D and that $f'(z) = \alpha f(z)$ for all $z \in D$, for some constant $\alpha \in \mathbb{C}$. Show that f must be of the form $f(z) = Ce^{\alpha z}$ for some constant $C \in \mathbb{C}$.