Engineering Probability and Statistics

Dispersion, Mean, Median, and Mode Values

If $X_1, X_2, \ldots, X_n$ represent the values of a random sample of n items or observations, the *arithmetic mean* of these items or observations, denoted $\overline{X}$, is defined as

$$\overline{X} = (1/n)(X_1 + X_2 + \ldots + X_n) = (1/n)\sum_{i=1}^{n} X_i$$

$\overline{X} \rightarrow \mu$ for sufficiently large values of n.

The *weighted arithmetic mean* is

$$\overline{X}_w = \frac{\sum w_i X_i}{\sum w_i}$$

where

X_i = the value of the ith observation, and

w_i = the weight applied to X_i.

The *variance* of the population is the *arithmetic mean* of the *squared deviations from the population mean*. If μ is the arithmetic mean of a discrete population of size N, the *population variance* is defined by

$$\sigma^2 = (1/N)\left[(X_1 - \mu)^2 + (X_2 - \mu)^2 + \ldots + (X_N - \mu)^2\right]$$

$$= (1/N)\sum_{i=1}^{N} (X_i - \mu)^2$$

Standard deviation formulas (assuming statistical independence) are

$$\sigma_{\text{population}} = \sqrt{(1/N)\sum (X_i - \mu)^2}$$

$$\sigma_{\text{sum}} = \sqrt{\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2}$$

$$\sigma_{\text{series}} = \sigma\sqrt{n}$$

$$\sigma_{\text{mean}} = \frac{\sigma}{\sqrt{n}}$$

$$\sigma_{\text{product}} = \sqrt{A^2\sigma_b^2 + B^2\sigma_a^2}$$

The *sample variance* is

$$s^2 = [1/(n-1)]\sum_{i=1}^{n} (X_i - \overline{X})^2$$

The *sample standard deviation* is

$$s = \sqrt{[1/(n-1)]\sum_{i=1}^{n} (X_i - \overline{X})^2}$$

The *sample coefficient of variation* = $CV = s/\overline{X}$

The *sample geometric mean* = $\sqrt[n]{X_1 X_2 X_3 \ldots X_n}$

The *sample root-mean-square value* = $\sqrt{(1/n)\sum X_i^2}$

When the discrete data are rearranged in increasing order and n is odd, the median is the value of the $\left(\frac{n+1}{2}\right)^{\text{th}}$ item

When n is even, the median is the average of the $\left(\frac{n}{2}\right)^{\text{th}}$ and $\left(\frac{n}{2} + 1\right)^{\text{th}}$ items.

The *mode* of a set of data is the value that occurs with greatest frequency.

The *sample range R* is the largest sample value minus the smallest sample value.

Permutations and Combinations

A *permutation* is a particular sequence of a given set of objects. A *combination* is the set itself without reference to order.

1. The number of different *permutations* of n distinct objects *taken r at a time* is

   $$P(n,r) = \frac{n!}{(n-r)!}$$

   nPr is an alternative notation for $P(n,r)$

2. The number of different *combinations* of n distinct objects *taken r at a time* is

   $$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{[r!(n-r)!]}$$

   nCr and $\binom{n}{r}$ are alternative notations for $C(n,r)$

3. The number of different *permutations* of n objects *taken n at a time*, given that n_i are of type i, where $i = 1, 2, \ldots, k$ and $\sum n_i = n$, is

   $$P(n; n_1, n_2, \ldots, n_k) = \frac{n!}{n_1!n_2!\ldots n_k!}$$

Sets

De Morgan's Law

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Associative Law

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive Law

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Laws of Probability

Property 1. General Character of Probability

The probability $P(E)$ of an event E is a real number in the range of 0 to 1. The probability of an impossible event is 0 and that of an event certain to occur is 1.

Property 2. Law of Total Probability

$$P(A + B) = P(A) + P(B) - P(A, B)$$

where

$P(A + B)$ = the probability that either A or B occur alone or that both occur together
$P(A)$ = the probability that A occurs
$P(B)$ = the probability that B occurs
$P(A, B)$ = the probability that both A and B occur simultaneously

Property 3. Law of Compound or Joint Probability

If neither $P(A)$ nor $P(B)$ is zero,

$$P(A, B) = P(A)P(B \mid A) = P(B)P(A \mid B)$$

where

$P(B \mid A)$ = the probability that B occurs given the fact that A has occurred

$P(A \mid B)$ = the probability that A occurs given the fact that B has occurred

If either $P(A)$ or $P(B)$ is zero, then $P(A, B) = 0$.

Bayes' Theorem

$$P\left(B_j \middle| A\right) = \frac{P\left(B_j\right)P\left(A \middle| B_j\right)}{\sum_{i=1}^{n} P\left(A \middle| B_i\right)P\left(B_i\right)}$$

where

$P\left(A_j\right)$ = the probability of event A_j within the population of A

$P\left(B_j\right)$ = the probability of event B_j within the population of B

Probability Functions, Distributions, and Expected Values

A random variable X has a probability associated with each of its possible values. The probability is termed a discrete probability if X can assume only discrete values, or

$$X = x_1, x_2, x_3, \ldots, x_n$$

The *discrete probability* of any single event, $X = x_i$, occurring is defined as $P(x_i)$ while the *probability mass function* of the random variable X is defined by

$$f(x_k) = P(X = x_k), k = 1, 2, ..., n$$

Probability Density Function

If X is continuous, the *probability density function, f,* is defined such that

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

Cumulative Distribution Functions

The *cumulative distribution function, F,* of a discrete random variable X that has a probability distribution described by $P(x_i)$ is defined as

$$F(x_m) = \sum_{k=1}^{m} P(x_k) = P(X \le x_m), m = 1, 2, \ldots, n$$

If X is continuous, the *cumulative distribution function, F,* is defined by

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

which implies that $F(a)$ is the probability that $X \le a$.

Expected Values

Let X be a discrete random variable having a probability mass function

$$f(x_k), k = 1, 2,..., n$$

The expected value of X is defined as

$$\mu = E[X] = \sum_{k=1}^{n} x_k f(x_k)$$

The variance of X is defined as

$$\sigma^2 = V[X] = \sum_{k=1}^{n} (x_k - \mu)^2 f(x_k)$$

Let X be a continuous random variable having a density function $f(X)$ and let $Y = g(X)$ be some general function. The expected value of Y is:

$$E[Y] = E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

The mean or expected value of the random variable X is now defined as

$$\mu = E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

while the variance is given by

$$\sigma^2 = V[X] = E\left[(X - \mu)^2\right] = E[x^2] - \mu^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx$$

The standard deviation is given by

$$\sigma = \sqrt{V[X]}$$

The coefficient of variation is defined as σ/μ.

Combinations of Random Variables

$$Y = a_1 X_1 + a_2 X_2 + \ldots + a_n X_n$$

The expected value of Y is:

$$\mu_y = E(Y) = a_1 E(X_1) + a_2 E(X_2) + \ldots + a_n E(X_n)$$

If the random variables are statistically *independent*, then the variance of Y is:

$$\sigma_y^2 = V(Y) = a_1^2 V(X_1) + a_2^2 V(X_2) + \ldots + a_n^2 V(X_n)$$
$$= a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 + \ldots + a_n^2 \sigma_n^2$$

Also, the standard deviation of Y is:

$$\sigma_y = \sqrt{\sigma_y^2}$$

When $Y = f(X_1, X_2,.., X_n)$ and X_i are independent, the standard deviation of Y is expressed as:

$$\sigma_y = \sqrt{\left(\frac{\partial f}{\partial X_1}\sigma_{X_1}\right)^2 + \left(\frac{\partial f}{\partial X_2}\sigma_{X_2}\right)^2 + \ldots + \left(\frac{\partial f}{\partial X_n}\sigma_{X_n}\right)^2}$$

Binomial Distribution

$P(x)$ is the probability that x successes will occur in n trials.

If p = probability of success and q = probability of failure = $1 - p$, then

$$P_n(x) = C(n,x)\, p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

where

x = 0, 1, 2, …, n

$C(n, x)$ = number of combinations

n, p = parameters

The variance is given by the form:

$\sigma^2 = npq$

Normal Distribution (Gaussian Distribution)

This is a unimodal distribution, the mode being $x = \mu$, with two points of inflection (each located at a distance σ to either side of the mode). The averages of n observations tend to become normally distributed as n increases. The variate x is said to be normally distributed if its density function $f(x)$ is given by an expression of the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where

μ = population mean

σ = standard deviation of the population

$-\infty \le x \le \infty$

When $\mu = 0$ and $\sigma^2 = \sigma = 1$, the distribution is called a *standardized* or *unit normal* distribution. Then

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \text{ where } -\infty \le x \le \infty.$$

A unit normal distribution table is included at the end of this section. In the table, the following notations are utilized:

$F(x)$ = area under the curve from $-\infty$ to x

$R(x)$ = area under the curve from x to ∞

$W(x)$ = area under the curve between $-x$ and x

$F(-x) = 1 - F(x)$

It should be noted that for any normal distribution with mean μ and standard deviation σ, the table for the unit normal distribution can be used by utilizing the following transformation:

$$z = \frac{x - \mu}{\sigma}$$

$f(x)$ then becomes $f(z)$, $F(x)$ becomes $F(z)$, etc.

<u>The Central Limit Theorem</u>

Let $X_1, X_2, ..., X_n$ be a sequence of independent and identically distributed random variables each having mean μ and variance σ^2. Then for large n, the Central Limit Theorem asserts that the sum

$Y = X_1 + X_2 + ... X_n$ is approximately normal.

$\mu_{\bar{y}} = \mu$

and the standard deviation

$$\sigma_{\bar{y}} = \frac{\sigma}{\sqrt{n}}$$

t-Distribution

Student's *t*-distribution has the probability density function given by:

$$f(t) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\,\Gamma\left(\frac{v}{2}\right)}\left(1+\frac{t^2}{v}\right)^{-\frac{v+1}{2}}$$

where

v = number of degrees of freedom

n = sample size

$v = n - 1$

Γ = gamma function

$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$

$-\infty \le t \le \infty$

A table later in this section gives the values of $t_{\alpha,\, v}$ for values of α and v. Note that, in view of the symmetry of the *t*-distribution, $t_{1-\alpha,v} = -t_{\alpha,v}$

The function for α follows:

$$\alpha = \int_{t_{\alpha,v}}^{\infty} f(t)\,dt$$

χ^2 - Distribution

If $Z_1, Z_2, ..., Z_n$ are independent unit normal random variables, then

$$\chi^2 = Z_1^2 + Z_2^2 + \ldots + Z_n^2$$

is said to have a chi-square distribution with n degrees of freedom.

A table at the end of this section gives values of $\chi^2_{\alpha,n}$ for selected values of α and n.

Gamma Function

$$\Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t}\,dt,\ n > 0$$

Propagation of Error

Measurement Error

Measurement error is defined as: *Measured quantity value minus a reference quantity value*. [Source: ISO JCGM 200:2012 definition 2.16]

Sources of errors in measurements arise from imperfections and disturbances in the measurement process, and added noise. One may model a measurement as:

$$x = x_{\text{ref}} + d_{\text{systematic}} + d_{\text{random}}$$

where x is the measurand (value being measured), x_{ref} is the reference value, $d_{\text{systematic}}$ is a disturbance from the measurement process such as a drift or bias, and d_{random} is a disturbance such as random noise.

Linear Combinations

In mathematics, a linear combination is an expression constructed from a set of terms by multiplying each term by a constant and adding the results (e.g., if z is a linear combination of x and y, then $z = ax+by$ where a and b are constants).

See the section "Combinations of Random Variables" for how variances and standard deviations of random variables combine.

Measurement Uncertainty

Measurement uncertainty is defined as: *A quantitative estimate of the range of values about the reported or measured value in which the true value is believed to lie.* [Source: ISO JCGM 200:2012, definition 2.26]

Given a desired state or measurement y, which is a function of different measured or available states x_i:

$$y = f(x_1, x_2, \ldots, x_n)$$

Given the individual states x_i and their standard deviations σ_{x_i}, and assuming that the different x_i are uncorrelated, the Kline-McClintock equation can be used to compute the expected standard uncertainty of y (σ_y) is:

$$\sigma_y = \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x_2}^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_{x_n}^2}$$

Expanded uncertainties are typically given at an approximately 95% level of confidence with a coverage factor of k = 2. This represents 95% of the area under a Normal probability distribution and is often called 2 sigma.

Linear Regression and Goodness of Fit

Least Squares

$$\hat{y} = \hat{a} + \hat{b}x$$

where

$$\hat{b} = S_{xy}/S_{xx}$$

$$\hat{a} = \overline{y} - \hat{b}\overline{x}$$

$$S_{xy} = \sum_{i=1}^{n} x_i y_i - (1/n)\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)$$

$$S_{xx} = \sum_{i=1}^{n} x_i^2 - (1/n)\left(\sum_{i=1}^{n} x_i\right)^2$$

$$\overline{y} = (1/n)\left(\sum_{i=1}^{n} y_i\right)$$

$$\overline{x} = (1/n)\left(\sum_{i=1}^{n} x_i\right)$$

where

n = sample size

Residual

$$e_i = y_i - \hat{y} = y_i - \left(\hat{a} + \hat{b}x_i\right)$$

Standard Error of Estimate $\left(S_e^2\right)$:

$$S_e^2 = \frac{S_{xx}S_{yy} - S_{xy}^2}{S_{xx}(n-2)} = MSE$$

where

$$S_{yy} = \sum_{i=1}^{n} y_i^2 - (1/n)\left(\sum_{i=1}^{n} y_i\right)^2$$

Confidence Interval for Intercept ($\hat{a}$):

$$\hat{a} \pm t_{\alpha/2,n-2}\sqrt{\left(\frac{1}{n} + \frac{\overline{x}^2}{S_{xx}}\right)MSE}$$

Confidence Interval for Slope ($\hat{b}$):

$$\hat{b} \pm t_{\alpha/2,n-2}\sqrt{\frac{MSE}{S_{xx}}}$$

Sample Correlation Coefficient (R) and Coefficient of Determination (R^2):

$$R = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$$

$$R^2 = \frac{S_{xy}^2}{S_{xx}S_{yy}}$$

Hypothesis Testing

Let a "dot" subscript indicate summation over the subscript. Thus:

$$y_{i\bullet} = \sum_{j=1}^{n} y_{ij} \qquad \text{and} \qquad y_{\bullet\bullet} = \sum_{i=1}^{a}\sum_{j=1}^{n} y_{ij}$$

One-Way Analysis of Variance (ANOVA)

Given independent random samples of size n_i from k populations, then:

$$\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(y_{ij}-\overline{y}_{\bullet\bullet}\right)^2 = \sum_{i=1}^{k} n_i\left(\overline{y}_{i\bullet}-\overline{y}_{\bullet\bullet}\right)^2 + \sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(y_{ij}-\overline{y}_{i\bullet}\right)^2$$

$$SS_{\text{total}} = SS_{\text{treatments}} + SS_{\text{error}}$$

If N = total number observations

$$N = \sum_{i=1}^{k} n_i, \ then$$

$$SS_{\text{total}} = \sum_{i=1}^{k}\sum_{j=1}^{n_i} y_{ij}^2 - \frac{y_{\bullet\bullet}^2}{N}$$

$$SS_{\text{treatments}} = \sum_{i=1}^{k}\frac{y_{i\bullet}^2}{n_i} - \frac{y_{\bullet\bullet}^2}{N}$$

$$SS_{\text{error}} = SS_{\text{total}} - SS_{\text{treatments}}$$

Montgomery, Douglas C., and George C. Runger, *Applied Statistics and Probability for Engineers*, 4 ed., New York: John Wiley and Sons, 2007.

Randomized Complete Block Design

For k treatments and b blocks

$$\sum_{i=1}^{k}\sum_{j=1}^{b}\left(y_{ij}-\overline{y}_{\bullet\bullet}\right)^2 = b\sum_{i=1}^{k}\left(\overline{y}_{i\bullet}-\overline{y}_{\bullet\bullet}\right)^2 + k\sum_{j=1}^{b}\left(\overline{y}_{\bullet j}-\overline{y}_{\bullet\bullet}\right)^2 + \sum_{i=1}^{k}\sum_{j=1}^{b}\left(\overline{y}_{ij}-\overline{y}_{\bullet j}-\overline{y}_{i\bullet}+\overline{y}_{\bullet\bullet}\right)^2$$

$$SS_{\text{total}} = SS_{\text{treatments}} + SS_{\text{blocks}} + SS_{\text{error}}$$

$$SS_{\text{total}} = \sum_{i=1}^{k}\sum_{j=1}^{b} y_{ij}^2 - \frac{y_{\bullet\bullet}^2}{kb}$$

$$SS_{\text{treatments}} = \frac{1}{b}\sum_{i=1}^{k} y_{i\bullet}^2 - \frac{y_{\bullet\bullet}^2}{bk}$$

$$SS_{blocks} = \frac{1}{k}\sum_{j=1}^{b} y_{\bullet j}^2 - \frac{y_{\bullet\bullet}^2}{bk}$$

$$SS_{\text{error}} = SS_{\text{total}} - SS_{\text{treatments}} - SS_{\text{blocks}}$$

Montgomery, Douglas C., and George C. Runger, *Applied Statistics and Probability for Engineers*, 4 ed., New York: John Wiley and Sons, 2007.

Two-Factor Factorial Designs

For a levels of Factor A, b levels of Factor B, and n repetitions per cell:

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(y_{ijk}-\overline{y}_{\bullet\bullet\bullet}\right)^2 = bn\sum_{i=1}^{a}\left(\overline{y}_{i\bullet\bullet}-\overline{y}_{\bullet\bullet\bullet}\right)^2 + an\sum_{j=1}^{b}\left(\overline{y}_{\bullet j\bullet}-\overline{y}_{\bullet\bullet\bullet}\right)^2$$

$$+\,n\sum_{i=1}^{a}\sum_{j=1}^{b}\left(\overline{y}_{ij\bullet}-\overline{y}_{i\bullet\bullet}-\overline{y}_{\bullet j\bullet}+\overline{y}_{\bullet\bullet\bullet}\right)^2 + \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(y_{ijk}-\overline{y}_{ij\bullet}\right)^2$$

$$SS_{\text{total}} = SS_{\text{A}} + SS_{\text{B}} + SS_{\text{AB}} + SS_{\text{error}}$$

$$SS_{\text{total}} = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk}^2 - \frac{y_{\bullet\bullet\bullet}^2}{abn}$$

$$SS_{\text{A}} = \sum_{i=1}^{a}\frac{y_{i\bullet\bullet}^2}{bn} - \frac{y_{\bullet\bullet\bullet}^2}{abn}$$

$$SS_{\text{B}} = \sum_{j=1}^{b}\frac{y_{\bullet j\bullet}^2}{an} - \frac{y_{\bullet\bullet\bullet}^2}{abn}$$

$$SS_{\text{AB}} = \sum_{i=1}^{a}\sum_{j=1}^{b}\frac{y_{ij\bullet}^2}{n} - \frac{y_{\bullet\bullet\bullet}^2}{abn} - SS_{\text{A}} - SS_{\text{B}}$$

$$SS_{\text{error}} = SS_{\text{T}} - SS_{\text{A}} - SS_{\text{B}} - SS_{\text{AB}}$$

Montgomery, Douglas C., and George C. Runger, *Applied Statistics and Probability for Engineers*, 4 ed., New York: John Wiley and Sons, 2007.

One-Way ANOVA Table

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F
Between Treatments	$k-1$	$SS_{\text{treatments}}$	$MST = \dfrac{SS_{\text{treatments}}}{k-1}$	$\dfrac{MST}{MSE}$
Error	$N-k$	SS_{error}	$MSE = \dfrac{SS_{\text{error}}}{N-k}$	
Total	$N-1$	SS_{total}		

Randomized Complete Block ANOVA Table

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F
Between Treatments	$k-1$	$SS_{\text{treatments}}$	$MST = \dfrac{SS_{\text{treatments}}}{k-1}$	$\dfrac{MST}{MSE}$
Between Blocks	$n-1$	SS_{blocks}	$MSB = \dfrac{SS_{\text{blocks}}}{n-1}$	$\dfrac{MSB}{MSE}$
Error	$(k-1)(n-1)$	SS_{error}	$MSE = \dfrac{SS_{\text{error}}}{(k-1)(n-1)}$	
Total	$N-1$	SS_{total}		

Two-Way Factorial ANOVA Table

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	*F*
A Treatments	$a-1$	SS_A	$MSA = \dfrac{SS_A}{a-1}$	$\dfrac{MSA}{MSE}$
B Treatments	$b-1$	SS_B	$MSB = \dfrac{SS_B}{b-1}$	$\dfrac{MSB}{MSE}$
AB Interaction	$(a-1)(b-1)$	SS_{AB}	$MSAB = \dfrac{SS_{AB}}{(a-1)(b-1)}$	$\dfrac{MSAB}{MSE}$
Error	$ab(n-1)$	SS_{error}	$MSE = \dfrac{SS_E}{ab(n-1)}$	
Total	$abn-1$	SS_{total}		

Consider an unknown parameter θ of a statistical distribution. Let the null hypothesis be

H_0: $\mu = \mu_0$

and let the alternative hypothesis be

H_1: $\mu \neq \mu_0$

Rejecting H_0 when it is true is known as a Type I error, while accepting H_0 when it is wrong is known as a Type II error. Furthermore, the probabilities of Type I and Type II errors are usually represented by the symbols α and β, respectively:

α = probability (Type I error)

β = probability (Type II error)

The probability of a Type I error is known as the level of significance of the test.

Table A. Tests on Means of Normal Distribution—Variance Known

Hypothesis	*Test Statistic*	*Criteria for Rejection*
H_0: $\mu = \mu_0$ H_1: $\mu \neq \mu_0$		$\lvert Z_0 \rvert > Z_{\alpha/2}$
H_0: $\mu = \mu_0$ H_1: $\mu < \mu_0$	$Z_0 \equiv \dfrac{\overline{X} - \mu_0}{\sigma / \sqrt{n}}$	$Z_0 < -Z_\alpha$
H_0: $\mu = \mu_0$ H_1: $\mu > \mu_0$		$Z_0 > Z_\alpha$
H_0: $\mu_1 - \mu_2 = \gamma$ H_1: $\mu_1 - \mu_2 \neq \gamma$		$\lvert Z_0 \rvert > Z_{\alpha/2}$
H_0: $\mu_1 - \mu_2 = \gamma$ H_1: $\mu_1 - \mu_2 < \gamma$	$Z_0 \equiv \dfrac{\overline{X_1} - \overline{X_2} - \gamma}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$	$Z_0 < -Z_\alpha$
H_0: $\mu_1 - \mu_2 = \gamma$ H_1: $\mu_1 - \mu_2 > \gamma$		$Z_0 > Z_\alpha$

Table B. Tests on Means of Normal Distribution—Variance Unknown

Hypothesis	*Test Statistic*	*Criteria for Rejection*
H_0: $\mu = \mu_0$ H_1: $\mu \neq \mu_0$		$\lvert t_0 \rvert > t_{\alpha/2,\, n-1}$
H_0: $\mu = \mu_0$ H_1: $\mu < \mu_0$	$t_0 = \dfrac{\overline{X} - \mu_0}{s / \sqrt{n}}$	$t_0 < -t_{\alpha,\, n-1}$
H_0: $\mu = \mu_0$ H_1: $\mu > \mu_0$		$t_0 > t_{\alpha,\, n-1}$
H_0: $\mu_1 - \mu_2 = \gamma$ H_1: $\mu_1 - \mu_2 \neq \gamma$	Variances equal: $t_0 = \dfrac{\overline{X_1} - \overline{X_2} - \gamma}{s_p \sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$ $v = n_1 + n_2 - 2$	$\lvert t_0 \rvert > t_{\alpha/2,\, v}$
H_0: $\mu_1 - \mu_2 = \gamma$ H_1: $\mu_1 - \mu_2 < \gamma$	Variances unequal: $t_0 = \dfrac{\overline{X_1} - \overline{X_2} - \gamma}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$	$t_0 < -t_{\alpha,\, v}$
H_0: $\mu_1 - \mu_2 = \gamma$ H_1: $\mu_1 - \mu_2 > \gamma$	$v = \dfrac{\left(\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}\right)^2}{\dfrac{\left(s_1^2/n_1\right)^2}{n_1 - 1} + \dfrac{\left(s_2^2/n_2\right)^2}{n_2 - 1}}$	$t_0 > t_{\alpha,\, v}$

In Table B, $s_p^2 = [(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2]/v$

Table C. Tests on Variances of Normal Distribution with Unknown Mean

Hypothesis	*Test Statistic*	*Criteria for Rejection*
H_0: $\sigma^2 = \sigma_0^2$ H_1: $\sigma^2 \neq \sigma_0^2$		$\chi_0^2 > \chi^2_{\alpha/2,\, n-1}$ or $\chi_0^2 < \chi^2_{1-\alpha/2,\, n-1}$
H_0: $\sigma^2 = \sigma_0^2$ H_1: $\sigma^2 < \sigma_0^2$	$\chi_0^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$	$\chi_0^2 < \chi^2_{1-\alpha,\, n-1}$
H_0: $\sigma^2 = \sigma_0^2$ H_1: $\sigma^2 > \sigma_0^2$		$\chi_0^2 > \chi^2_{\alpha,\, n-1}$
H_0: $\sigma_1^2 = \sigma_2^2$ H_1: $\sigma_1^2 \neq \sigma_2^2$	$F_0 = \dfrac{s_1^2}{s_2^2}$	$F_0 > F_{\alpha/2,\, n_1-1,\, n_2-1}$ $F_0 < F_{1-\alpha/2,\, n_1-1,\, n_2-1}$
H_0: $\sigma_1^2 = \sigma_2^2$ H_1: $\sigma_1^2 < \sigma_2^2$	$F_0 = \dfrac{s_2^2}{s_1^2}$	$F_0 > F_{\alpha,\, n_2-1,\, n_1-1}$
H_0: $\sigma_1^2 = \sigma_2^2$ H_1: $\sigma_1^2 > \sigma_2^2$	$F_0 = \dfrac{s_1^2}{s_2^2}$	$F_0 > F_{\alpha,\, n_1-1,\, n_2-1}$

Assume that the values of α and β are given. The sample size can be obtained from the following relationships. In (A) and (B), μ_1 is the value assumed to be the true mean.

(A) $H_0: \mu = \mu_0;\ H_1: \mu \neq \mu_0$

$$\beta = \Phi\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + Z_{a/2}\right) - \Phi\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} - Z_{a/2}\right)$$

An approximate result is

$$n \simeq \frac{\left(Z_{a/2} + Z_b\right)^2 \sigma^2}{\left(\mu_1 - \mu_0\right)^2}$$

(B) $H_0: \mu = \mu_0;\ H_1: \mu > \mu_0$

$$\beta = \Phi\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + Z_a\right)$$

$$n = \frac{\left(Z_a + Z_b\right)^2 \sigma^2}{\left(\mu_1 - \mu_0\right)^2}$$

Confidence Intervals, Sample Distributions and Sample Size

Confidence Interval for the Mean μ of a Normal Distribution

(A) Standard deviation σ is known

$$\overline{X} - Z_{a/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \overline{X} + Z_{a/2}\frac{\sigma}{\sqrt{n}}$$

(B) Standard deviation σ is not known

$$\overline{X} - t_{a/2}\frac{s}{\sqrt{n}} \le \mu \le \overline{X} + t_{a/2}\frac{s}{\sqrt{n}}$$

where $t_{a/2}$ corresponds to $n - 1$ degrees of freedom.

Confidence Interval for the Difference Between Two Means μ_1 and μ_2

(A) Standard deviations σ_1 and σ_2 known

$$\overline{X}_1-\overline{X}_2-Z_{a/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}\le \mu_1-\mu_2\le \overline{X}_1-\overline{X}_2+Z_{a/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}$$

(B) Standard deviations σ_1 and σ_2 are not known

$$\overline{X}_1-\overline{X}_2-t_{a/2}\sqrt{\frac{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\left[(n_1-1)s_1^2+(n_2-1)s_2^2\right]}{n_1+n_2-2}}\le \mu_1-\mu_2\le \overline{X}_1-\overline{X}_2+t_{a/2}\sqrt{\frac{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\left[(n_1-1)s_1^2+(n_2-1)s_2^2\right]}{n_1+n_2-2}}$$

where $t_{a/2}$ corresponds to n_1+n_2-2 degrees of freedom.

Confidence Intervals for the Variance σ^2 of a Normal Distribution

$$\frac{(n-1)s^2}{x^2_{\alpha/2,n-1}}\le\sigma^2\le\frac{(n-1)s^2}{x^2_{1-\alpha/2,n-1}}$$

Sample Size

$$z=\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\qquad n=\left[\frac{z_{\alpha/2}\sigma}{\overline{x}-\mu}\right]^2$$

Test Statistics

The following definitions apply.

$$Z_{\text{var}}=\frac{\overline{X}-\mu_o}{\frac{\sigma}{\sqrt{n}}}$$

$$t_{\text{var}}=\frac{\overline{X}-\mu_o}{\frac{s}{\sqrt{n}}}$$

where

Z_{var} = standard normal Z score
t_{var} = sample distribution test statistic
σ = standard deviation
μ_o = population mean
$\overline{X}$ = hypothesized mean or sample mean
n = sample size
s = computed sample standard deviation

The Z score is applicable when the standard deviation (s) is known. The test statistic is applicable when the standard deviation (s) is computed at time of sampling.

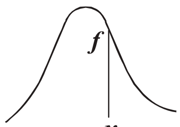
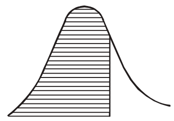
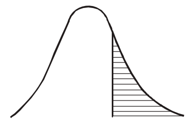
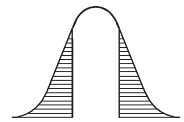
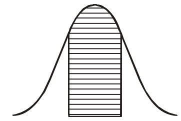
Z_α corresponds to the appropriate probability under the normal probability curve for a given Z_{var}.

$t_{\alpha,\,n-1}$ corresponds to the appropriate probability under the t distribution with $n-1$ degrees of freedom for a given t_{var}.

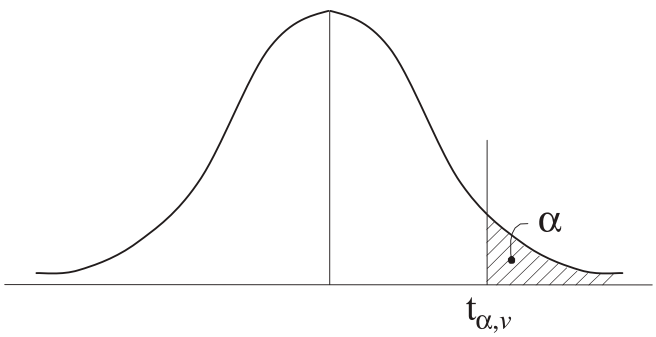
Values of $Z_{\alpha/2}$

Confidence Interval	$Z_{\alpha/2}$
80%	1.2816
90%	1.6449
95%	1.9600
96%	2.0537
98%	2.3263
99%	2.5758

Unit Normal Distribution ($\mu = 0$, $\sigma = 1$)

	 x	 x	 x	 $-x$ x	 $-x$ x
x	$f(x)$	$F(x)$	$R(x)$	$2R(x)$	$W(x)$
0.0	0.3989	0.5000	0.5000	1.0000	0.0000
0.1	0.3970	0.5398	0.4602	0.9203	0.0797
0.2	0.3910	0.5793	0.4207	0.8415	0.1585
0.3	0.3814	0.6179	0.3821	0.7642	0.2358
0.4	0.3683	0.6554	0.3446	0.6892	0.3108
0.5	0.3521	0.6915	0.3085	0.6171	0.3829
0.6	0.3332	0.7257	0.2743	0.5485	0.4515
0.7	0.3123	0.7580	0.2420	0.4839	0.5161
0.8	0.2897	0.7881	0.2119	0.4237	0.5763
0.9	0.2661	0.8159	0.1841	0.3681	0.6319
1.0	0.2420	0.8413	0.1587	0.3173	0.6827
1.1	0.2179	0.8643	0.1357	0.2713	0.7287
1.2	0.1942	0.8849	0.1151	0.2301	0.7699
1.3	0.1714	0.9032	0.0968	0.1936	0.8064
1.4	0.1497	0.9192	0.0808	0.1615	0.8385
1.5	0.1295	0.9332	0.0668	0.1336	0.8664
1.6	0.1109	0.9452	0.0548	0.1096	0.8904
1.7	0.0940	0.9554	0.0446	0.0891	0.9109
1.8	0.0790	0.9641	0.0359	0.0719	0.9281
1.9	0.0656	0.9713	0.0287	0.0574	0.9426
2.0	0.0540	0.9772	0.0228	0.0455	0.9545
2.1	0.0440	0.9821	0.0179	0.0357	0.9643
2.2	0.0355	0.9861	0.0139	0.0278	0.9722
2.3	0.0283	0.9893	0.0107	0.0214	0.9786
2.4	0.0224	0.9918	0.0082	0.0164	0.9836
2.5	0.0175	0.9938	0.0062	0.0124	0.9876
2.6	0.0136	0.9953	0.0047	0.0093	0.9907
2.7	0.0104	0.9965	0.0035	0.0069	0.9931
2.8	0.0079	0.9974	0.0026	0.0051	0.9949
2.9	0.0060	0.9981	0.0019	0.0037	0.9963
3.0	0.0044	0.9987	0.0013	0.0027	0.9973
Fractiles					
1.2816	0.1755	0.9000	0.1000	0.2000	0.8000
1.6449	0.1031	0.9500	0.0500	0.1000	0.9000
1.9600	0.0584	0.9750	0.0250	0.0500	0.9500
2.0537	0.0484	0.9800	0.0200	0.0400	0.9600
2.3263	0.0267	0.9900	0.0100	0.0200	0.9800
2.5758	0.0145	0.9950	0.0050	0.0100	0.9900

Student's *t*-Distribution

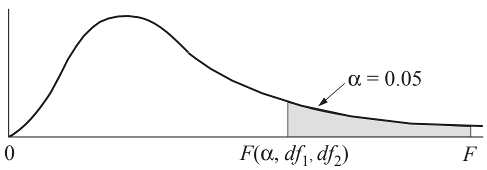


VALUES OF $t_{\alpha,v}$

v	α								*v*
	0.25	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**	
1	1.000	1.376	1.963	3.078	6.314	12.706	31.821	63.657	1
2	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	2
3	0.765	0.978	1.350	1.638	2.353	3.182	4.541	5.841	3
4	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	4
5	**0.727**	**0.920**	**1.156**	**1.476**	**2.015**	**2.571**	**3.365**	**4.032**	**5**
6	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	6
7	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	7
8	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	8
9	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	9
10	**0.700**	**0.879**	**1.093**	**1.372**	**1.812**	**2.228**	**2.764**	**3.169**	**10**
11	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	11
12	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	12
13	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	13
14	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	14
15	**0.691**	**0.866**	**1.074**	**1.341**	**1.753**	**2.131**	**2.602**	**2.947**	**15**
16	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	16
17	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	17
18	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	18
19	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	19
20	**0.687**	**0.860**	**1.064**	**1.325**	**1.725**	**2.086**	**2.528**	**2.845**	**20**
21	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	21
22	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	22
23	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	23
24	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	24
25	**0.684**	**0.856**	**1.058**	**1.316**	**1.708**	**2.060**	**2.485**	**2.787**	**25**
26	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	26
27	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	27
28	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	28
29	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	29
30	**0.683**	**0.854**	**1.055**	**1.310**	**1.697**	**2.042**	**2.457**	**2.750**	**30**
∞	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	∞

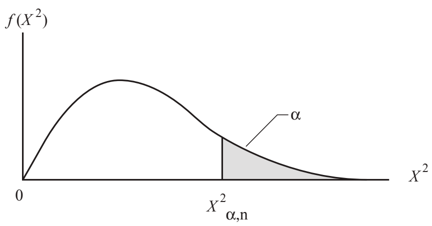
CRITICAL VALUES OF THE *F* DISTRIBUTION – TABLE

For a particular combination of numerator and denominator degrees of freedom, entry represents the critical values of *F* corresponding to a specified upper tail area (α).



Denominator df_2	Numerator df_1 1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	253.3	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	**6.61**	**5.79**	**5.41**	**5.19**	**5.05**	**4.95**	**4.88**	**4.82**	**4.77**	**4.74**	**4.68**	**4.62**	**4.56**	**4.53**	**4.50**	**4.46**	**4.43**	**4.40**	**4.36**
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	**4.96**	**4.10**	**3.71**	**3.48**	**3.33**	**3.22**	**3.14**	**3.07**	**3.02**	**2.98**	**2.91**	**2.85**	**2.77**	**2.74**	**2.70**	**2.66**	**2.62**	**2.58**	**2.54**
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	**4.54**	**3.68**	**3.29**	**3.06**	**2.90**	**2.79**	**2.71**	**2.64**	**2.59**	**2.54**	**2.48**	**2.40**	**2.33**	**2.29**	**2.25**	**2.20**	**2.16**	**2.11**	**2.07**
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	**4.35**	**3.49**	**3.10**	**2.87**	**2.71**	**2.60**	**2.51**	**2.45**	**2.39**	**2.35**	**2.28**	**2.20**	**2.12**	**2.08**	**2.04**	**1.99**	**1.95**	**1.90**	**1.84**
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	**4.24**	**3.39**	**2.99**	**2.76**	**2.60**	**2.49**	**2.40**	**2.34**	**2.28**	**2.24**	**2.16**	**2.09**	**2.01**	**1.96**	**1.92**	**1.87**	**1.82**	**1.77**	**1.71**
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	**4.17**	**3.32**	**2.92**	**2.69**	**2.53**	**2.42**	**2.33**	**2.27**	**2.21**	**2.16**	**2.09**	**2.01**	**1.93**	**1.89**	**1.84**	**1.79**	**1.74**	**1.68**	**1.62**
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

CRITICAL VALUES OF X^2 DISTRIBUTION



Degrees of Freedom	$X^2_{.995}$	$X^2_{.990}$	$X^2_{.975}$	$X^2_{.950}$	$X^2_{.900}$	$X^2_{.100}$	$X^2_{.050}$	$X^2_{.025}$	$X^2_{.010}$	$X^2_{.005}$
1	0.0000393	0.0001571	0.0009821	0.0039321	0.0157908	2.70554	3.84146	5.02389	6.63490	7.87944
2	0.0100251	0.0201007	0.0506356	0.102587	0.210720	4.60517	5.99147	7.37776	9.21034	10.5966
3	0.0717212	0.114832	0.215795	0.351846	0.584375	6.25139	7.81473	9.34840	11.3449	12.8381
4	0.206990	0.297110	0.484419	0.710721	1.063623	7.77944	9.48773	11.1433	13.2767	14.8602
5	0.411740	0.554300	0.831211	1.145476	1.61031	9.23635	11.0705	12.8325	15.0863	16.7496
6	0.675727	0.872085	1.237347	1.63539	2.20413	10.6446	12.5916	14.4494	16.8119	18.5476
7	0.989265	1.239043	1.68987	2.16735	2.83311	12.0170	14.0671	16.0128	18.4753	20.2777
8	1.344419	1.646482	2.17973	2.73264	3.48954	13.3616	15.5073	17.5346	20.0902	21.9550
9	1.734926	2.087912	2.70039	3.32511	4.16816	14.6837	16.9190	19.0228	21.6660	23.5893
10	2.15585	2.55821	3.24697	3.94030	4.86518	15.9871	18.3070	20.4831	23.2093	25.1882
11	2.60321	3.05347	3.81575	4.57481	5.57779	17.2750	19.6751	21.9200	24.7250	26.7569
12	3.07382	3.57056	4.40379	5.22603	6.30380	18.5494	21.0261	23.3367	26.2170	28.2995
13	3.56503	4.10691	5.00874	5.89186	7.04150	19.8119	22.3621	24.7356	27.6883	29.8194
14	4.07468	4.66043	5.62872	6.57063	7.78953	21.0642	23.6848	26.1190	29.1413	31.3193
15	4.60094	5.22935	6.26214	7.26094	8.54675	22.3072	24.9958	27.4884	30.5779	32.8013
16	5.14224	5.81221	6.90766	7.96164	9.31223	23.5418	26.2962	28.8454	31.9999	34.2672
17	5.69724	6.40776	7.56418	8.67176	10.0852	24.7690	27.5871	30.1910	33.4087	35.7185
18	6.26481	7.01491	8.23075	9.39046	10.8649	25.9894	28.8693	31.5264	34.8053	37.1564
19	6.84398	7.63273	8.90655	10.1170	11.6509	27.2036	30.1435	32.8523	36.1908	38.5822
20	7.43386	8.26040	9.59083	10.8508	12.4426	28.4120	31.4104	34.1696	37.5662	39.9968
21	8.03366	8.89720	10.28293	11.5913	13.2396	29.6151	32.6705	35.4789	38.9321	41.4010
22	8.64272	9.54249	10.9823	12.3380	14.0415	30.8133	33.9244	36.7807	40.2894	42.7956
23	9.26042	10.19567	11.6885	13.0905	14.8479	32.0069	35.1725	38.0757	41.6384	44.1813
24	9.88623	10.8564	12.4011	13.8484	15.6587	33.1963	36.4151	39.3641	42.9798	45.5585
25	10.5197	11.5240	13.1197	14.6114	16.4734	34.3816	37.6525	40.6465	44.3141	46.9278
26	11.1603	12.1981	13.8439	15.3791	17.2919	35.5631	38.8852	41.9232	45.6417	48.2899
27	11.8076	12.8786	14.5733	16.1513	18.1138	36.7412	40.1133	43.1944	46.9630	49.6449
28	12.4613	13.5648	15.3079	16.9279	18.9392	37.9159	41.3372	44.4607	48.2782	50.9933
29	13.1211	14.2565	16.0471	17.7083	19.7677	39.0875	42.5569	45.7222	49.5879	52.3356
30	13.7867	14.9535	16.7908	18.4926	20.5992	40.2560	43.7729	46.9792	50.8922	53.6720
40	20.7065	22.1643	24.4331	26.5093	29.0505	51.8050	55.7585	59.3417	63.6907	66.7659
50	27.9907	29.7067	32.3574	34.7642	37.6886	63.1671	67.5048	71.4202	76.1539	79.4900
60	35.5346	37.4848	40.4817	43.1879	46.4589	74.3970	79.0819	83.2976	88.3794	91.9517
70	43.2752	45.4418	48.7576	51.7393	55.3290	85.5271	90.5312	95.0231	100.425	104.215
80	51.1720	53.5400	57.1532	60.3915	64.2778	96.5782	101.879	106.629	112.329	116.321
90	59.1963	61.7541	65.6466	69.1260	73.2912	107.565	113.145	118.136	124.116	128.299
100	67.3276	70.0648	74.2219	77.9295	82.3581	118.498	124.342	129.561	135.807	140.169

Source: Thompson, C. M., "Tables of the Percentage Points of the X^2–Distribution," *Biometrika,* ©1941, 32, 188-189. Reproduced by permission of Oxford University Press.

Cumulative Binomial Probabilities $P(X \leq x)$

		P										
n	*x*	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
1	0	0.9000	0.8000	0.7000	0.6000	0.5000	0.4000	0.3000	0.2000	0.1000	0.0500	0.0100
2	0	0.8100	0.6400	0.4900	0.3600	0.2500	0.1600	0.0900	0.0400	0.0100	0.0025	0.0001
	1	0.9900	0.9600	0.9100	0.8400	0.7500	0.6400	0.5100	0.3600	0.1900	0.0975	0.0199
3	0	0.7290	0.5120	0.3430	0.2160	0.1250	0.0640	0.0270	0.0080	0.0010	0.0001	0.0000
	1	0.9720	0.8960	0.7840	0.6480	0.5000	0.3520	0.2160	0.1040	0.0280	0.0073	0.0003
	2	0.9990	0.9920	0.9730	0.9360	0.8750	0.7840	0.6570	0.4880	0.2710	0.1426	0.0297
4	0	0.6561	0.4096	0.2401	0.1296	0.0625	0.0256	0.0081	0.0016	0.0001	0.0000	0.0000
	1	0.9477	0.8192	0.6517	0.4752	0.3125	0.1792	0.0837	0.0272	0.0037	0.0005	0.0000
	2	0.9963	0.9728	0.9163	0.8208	0.6875	0.5248	0.3483	0.1808	0.0523	0.0140	0.0006
	3	0.9999	0.9984	0.9919	0.9744	0.9375	0.8704	0.7599	0.5904	0.3439	0.1855	0.0394
5	0	0.5905	0.3277	0.1681	0.0778	0.0313	0.0102	0.0024	0.0003	0.0000	0.0000	0.0000
	1	0.9185	0.7373	0.5282	0.3370	0.1875	0.0870	0.0308	0.0067	0.0005	0.0000	0.0000
	2	0.9914	0.9421	0.8369	0.6826	0.5000	0.3174	0.1631	0.0579	0.0086	0.0012	0.0000
	3	0.9995	0.9933	0.9692	0.9130	0.8125	0.6630	0.4718	0.2627	0.0815	0.0226	0.0010
	4	1.0000	0.9997	0.9976	0.9898	0.9688	0.9222	0.8319	0.6723	0.4095	0.2262	0.0490
6	0	0.5314	0.2621	0.1176	0.0467	0.0156	0.0041	0.0007	0.0001	0.0000	0.0000	0.0000
	1	0.8857	0.6554	0.4202	0.2333	0.1094	0.0410	0.0109	0.0016	0.0001	0.0000	0.0000
	2	0.9842	0.9011	0.7443	0.5443	0.3438	0.1792	0.0705	0.0170	0.0013	0.0001	0.0000
	3	0.9987	0.9830	0.9295	0.8208	0.6563	0.4557	0.2557	0.0989	0.0159	0.0022	0.0000
	4	0.9999	0.9984	0.9891	0.9590	0.8906	0.7667	0.5798	0.3446	0.1143	0.0328	0.0015
	5	1.0000	0.9999	0.9993	0.9959	0.9844	0.9533	0.8824	0.7379	0.4686	0.2649	0.0585
7	0	0.4783	0.2097	0.0824	0.0280	0.0078	0.0016	0.0002	0.0000	0.0000	0.0000	0.0000
	1	0.8503	0.5767	0.3294	0.1586	0.0625	0.0188	0.0038	0.0004	0.0000	0.0000	0.0000
	2	0.9743	0.8520	0.6471	0.4199	0.2266	0.0963	0.0288	0.0047	0.0002	0.0000	0.0000
	3	0.9973	0.9667	0.8740	0.7102	0.5000	0.2898	0.1260	0.0333	0.0027	0.0002	0.0000
	4	0.9998	0.9953	0.9712	0.9037	0.7734	0.5801	0.3529	0.1480	0.0257	0.0038	0.0000
	5	1.0000	0.9996	0.9962	0.9812	0.9375	0.8414	0.6706	0.4233	0.1497	0.0444	0.0020
	6	1.0000	1.0000	0.9998	0.9984	0.9922	0.9720	0.9176	0.7903	0.5217	0.3017	0.0679
8	0	0.4305	0.1678	0.0576	0.0168	0.0039	0.0007	0.0001	0.0000	0.0000	0.0000	0.0000
	1	0.8131	0.5033	0.2553	0.1064	0.0352	0.0085	0.0013	0.0001	0.0000	0.0000	0.0000
	2	0.9619	0.7969	0.5518	0.3154	0.1445	0.0498	0.0113	0.0012	0.0000	0.0000	0.0000
	3	0.9950	0.9437	0.8059	0.5941	0.3633	0.1737	0.0580	0.0104	0.0004	0.0000	0.0000
	4	0.9996	0.9896	0.9420	0.8263	0.6367	0.4059	0.1941	0.0563	0.0050	0.0004	0.0000
	5	1.0000	0.9988	0.9887	0.9502	0.8555	0.6846	0.4482	0.2031	0.0381	0.0058	0.0001
	6	1.0000	0.9999	0.9987	0.9915	0.9648	0.8936	0.7447	0.4967	0.1869	0.0572	0.0027
	7	1.0000	1.0000	0.9999	0.9993	0.9961	0.9832	0.9424	0.8322	0.5695	0.3366	0.0773
9	0	0.3874	0.1342	0.0404	0.0101	0.0020	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.7748	0.4362	0.1960	0.0705	0.0195	0.0038	0.0004	0.0000	0.0000	0.0000	0.0000
	2	0.9470	0.7382	0.4628	0.2318	0.0898	0.0250	0.0043	0.0003	0.0000	0.0000	0.0000
	3	0.9917	0.9144	0.7297	0.4826	0.2539	0.0994	0.0253	0.0031	0.0001	0.0000	0.0000
	4	0.9991	0.9804	0.9012	0.7334	0.5000	0.2666	0.0988	0.0196	0.0009	0.0000	0.0000
	5	0.9999	0.9969	0.9747	0.9006	0.7461	0.5174	0.2703	0.0856	0.0083	0.0006	0.0000
	6	1.0000	0.9997	0.9957	0.9750	0.9102	0.7682	0.5372	0.2618	0.0530	0.0084	0.0001
	7	1.0000	1.0000	0.9996	0.9962	0.9805	0.9295	0.8040	0.5638	0.2252	0.0712	0.0034
	8	1.0000	1.0000	1.0000	0.9997	0.9980	0.9899	0.9596	0.8658	0.6126	0.3698	0.0865

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Cumulative Binomial Probabilities ***P*(*X* ≤ *x*) (*continued*)**

		P										
n	*x*	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
10	0	0.3487	0.1074	0.0282	0.0060	0.0010	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.7361	0.3758	0.1493	0.0464	0.0107	0.0017	0.0001	0.0000	0.0000	0.0000	0.0000
	2	0.9298	0.6778	0.3828	0.1673	0.0547	0.0123	0.0016	0.0001	0.0000	0.0000	0.0000
	3	0.9872	0.8791	0.6496	0.3823	0.1719	0.0548	0.0106	0.0009	0.0000	0.0000	0.0000
	4	0.9984	0.9672	0.8497	0.6331	0.3770	0.1662	0.0473	0.0064	0.0001	0.0000	0.0000
	5	0.9999	0.9936	0.9527	0.8338	0.6230	0.3669	0.1503	0.0328	0.0016	0.0001	0.0000
	6	1.0000	0.9991	0.9894	0.9452	0.8281	0.6177	0.3504	0.1209	0.0128	0.0010	0.0000
	7	1.0000	0.9999	0.9984	0.9877	0.9453	0.8327	0.6172	0.3222	0.0702	0.0115	0.0001
	8	1.0000	1.0000	0.9999	0.9983	0.9893	0.9536	0.8507	0.6242	0.2639	0.0861	0.0043
	9	1.0000	1.0000	1.0000	0.9999	0.9990	0.9940	0.9718	0.8926	0.6513	0.4013	0.0956
15	0	0.2059	0.0352	0.0047	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.5490	0.1671	0.0353	0.0052	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.8159	0.3980	0.1268	0.0271	0.0037	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.9444	0.6482	0.2969	0.0905	0.0176	0.0019	0.0001	0.0000	0.0000	0.0000	0.0000
	4	0.9873	0.8358	0.5155	0.2173	0.0592	0.0093	0.0007	0.0000	0.0000	0.0000	0.0000
	5	0.9978	0.9389	0.7216	0.4032	0.1509	0.0338	0.0037	0.0001	0.0000	0.0000	0.0000
	6	0.9997	0.9819	0.8689	0.6098	0.3036	0.0950	0.0152	0.0008	0.0000	0.0000	0.0000
	7	1.0000	0.9958	0.9500	0.7869	0.5000	0.2131	0.0500	0.0042	0.0000	0.0000	0.0000
	8	1.0000	0.9992	0.9848	0.9050	0.6964	0.3902	0.1311	0.0181	0.0003	0.0000	0.0000
	9	1.0000	0.9999	0.9963	0.9662	0.8491	0.5968	0.2784	0.0611	0.0022	0.0001	0.0000
	10	1.0000	1.0000	0.9993	0.9907	0.9408	0.7827	0.4845	0.1642	0.0127	0.0006	0.0000
	11	1.0000	1.0000	0.9999	0.9981	0.9824	0.9095	0.7031	0.3518	0.0556	0.0055	0.0000
	12	1.0000	1.0000	1.0000	0.9997	0.9963	0.9729	0.8732	0.6020	0.1841	0.0362	0.0004
	13	1.0000	1.0000	1.0000	1.0000	0.9995	0.9948	0.9647	0.8329	0.4510	0.1710	0.0096
	14	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9953	0.9648	0.7941	0.5367	0.1399
20	0	0.1216	0.0115	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.3917	0.0692	0.0076	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.6769	0.2061	0.0355	0.0036	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.8670	0.4114	0.1071	0.0160	0.0013	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	4	0.9568	0.6296	0.2375	0.0510	0.0059	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
	5	0.9887	0.8042	0.4164	0.1256	0.0207	0.0016	0.0000	0.0000	0.0000	0.0000	0.0000
	6	0.9976	0.9133	0.6080	0.2500	0.0577	0.0065	0.0003	0.0000	0.0000	0.0000	0.0000
	7	0.9996	0.9679	0.7723	0.4159	0.1316	0.0210	0.0013	0.0000	0.0000	0.0000	0.0000
	8	0.9999	0.9900	0.8867	0.5956	0.2517	0.0565	0.0051	0.0001	0.0000	0.0000	0.0000
	9	1.0000	0.9974	0.9520	0.7553	0.4119	0.1275	0.0171	0.0006	0.0000	0.0000	0.0000
	10	1.0000	0.9994	0.9829	0.8725	0.5881	0.2447	0.0480	0.0026	0.0000	0.0000	0.0000
	11	1.0000	0.9999	0.9949	0.9435	0.7483	0.4044	0.1133	0.0100	0.0001	0.0000	0.0000
	12	1.0000	1.0000	0.9987	0.9790	0.8684	0.5841	0.2277	0.0321	0.0004	0.0000	0.0000
	13	1.0000	1.0000	0.9997	0.9935	0.9423	0.7500	0.3920	0.0867	0.0024	0.0000	0.0000
	14	1.0000	1.0000	1.0000	0.9984	0.9793	0.8744	0.5836	0.1958	0.0113	0.0003	0.0000
	15	1.0000	1.0000	1.0000	0.9997	0.9941	0.9490	0.7625	0.3704	0.0432	0.0026	0.0000
	16	1.0000	1.0000	1.0000	1.0000	0.9987	0.9840	0.8929	0.5886	0.1330	0.0159	0.0000
	17	1.0000	1.0000	1.0000	1.0000	0.9998	0.9964	0.9645	0.7939	0.3231	0.0755	0.0010
	18	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9924	0.9308	0.6083	0.2642	0.0169
	19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9992	0.9885	0.8784	0.6415	0.1821

Montgomery, Douglas C., and George C. Runger, *Applied Statistics and Probability for Engineers*, 4 ed., New York: John Wiley and Sons, 2007.

Statistical Quality Control

Average and Range Charts

n	A_2	D_3	D_4
2	1.880	0	3.268
3	1.023	0	2.574
4	0.729	0	2.282
5	0.577	0	2.114
6	0.483	0	2.004
7	0.419	0.076	1.924
8	0.373	0.136	1.864
9	0.337	0.184	1.816
10	0.308	0.223	1.777

X_i = an individual observation

n = the sample size of a group

k = the number of groups

R = (range) the difference between the largest and smallest observations in a sample of size n.

$$\overline{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

$$\overline{\overline{X}} = \frac{\overline{X}_1 + \overline{X}_2 + \ldots + \overline{X}_k}{k}$$

$$\overline{R} = \frac{R_1 + R_2 + \ldots + R_k}{k}$$

The R Chart formulas are:

$$CL_R = \overline{R}$$

$$UCL_R = D_4\overline{R}$$

$$LCL_R = D_3\overline{R}$$

The $\overline{X}$ Chart formulas are:

$$CL_X = \overline{\overline{X}}$$

$$UCL_X = \overline{\overline{X}} + A_2\overline{R}$$

$$LCL_X = \overline{\overline{X}} - A_2\overline{R}$$

Standard Deviation Charts

n	A_3	B_3	B_4
2	2.659	0	3.267
3	1.954	0	2.568
4	1.628	0	2.266
5	1.427	0	2.089
6	1.287	0.030	1.970
7	1.182	0.119	1.882
8	1.099	0.185	1.815
9	1.032	0.239	1.761
10	0.975	0.284	1.716

$$UCL_X = \overline{\overline{X}} + A_3\overline{S}$$

$$CL_X = \overline{\overline{X}}$$

$$LCL_X = \overline{\overline{X}} - A_3\overline{S}$$

$$UCL_S = B_4\overline{S}$$

$$CL_S = \overline{S}$$

$$LCL_S = B_3\overline{S}$$

Approximations

The following table and equations may be used to generate initial approximations of the items indicated.

n	c_4	d_2	d_3
2	0.7979	1.128	0.853
3	0.8862	1.693	0.888
4	0.9213	2.059	0.880
5	0.9400	2.326	0.864
6	0.9515	2.534	0.848
7	0.9594	2.704	0.833
8	0.9650	2.847	0.820
9	0.9693	2.970	0.808
10	0.9727	3.078	0.797

$$\hat{\sigma} = \overline{R}/d_2$$
$$\hat{\sigma} = \overline{S}/c_4$$
$$\sigma_R = d_3\hat{\sigma}$$
$$\sigma_S = \hat{\sigma}\sqrt{1 - c_4^2}$$

where

$\hat{\sigma}$ = an estimate of σ

σ_R = an estimate of the standard deviation of the ranges of the samples

σ_S = an estimate of the standard deviation of the standard deviations of the samples

Tests for Out of Control

1. A single point falls outside the (three sigma) control limits.
2. Two out of three successive points fall on the same side of and more than two sigma units from the center line.
3. Four out of five successive points fall on the same side of and more than one sigma unit from the center line.
4. Eight successive points fall on the same side of the center line.

Probability and Density Functions: Means and Variances

Variable	Equation	Mean	Variance
Binomial Coefficient	$\binom{n}{x} = \frac{n!}{x!(n-x)!}$		
Binomial	$b(x;n,p) = \binom{n}{x} p^x (1-p)^{n-x}$	np	$np(1-p)$
Hyper Geometric	$h(x;n,r,N) = \binom{r}{x} \frac{\binom{N-r}{n-x}}{\binom{N}{n}}$	$\frac{nr}{N}$	$\frac{r(N-r)n(N-n)}{N^2(N-1)}$
Poisson	$f(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$	λ	λ
Geometric	$g(x;p) = p(1-p)^{x-1}$	$1/p$	$(1-p)/p^2$
Negative Binomial	$f(y;r,p) = \binom{y+r-1}{r-1} p^r (1-p)^y$	r/p	$r(1-p)/p^2$
Multinomial	$f(x_1,\ldots,x_k) = \frac{n!}{x_1!,\ldots,x_k!} p_1^{x_1} \ldots p_k^{x_k}$	np_i	$np_i(1-p_i)$
Uniform	$f(x) = 1/(b-a)$	$(a+b)/2$	$(b-a)^2/12$
Gamma	$f(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}; \quad \alpha > 0, \beta > 0$	$\alpha\beta$	$\alpha\beta^2$
Exponential	$f(x) = \frac{1}{\beta} e^{-x/\beta}$	β	β^2
Weibull	$f(x) = \frac{\alpha}{\beta} x^{\alpha-1} e^{-x^\alpha/\beta}$	$\beta^{1/\alpha}\Gamma[(\alpha+1)/\alpha]$	$\beta^{2/\alpha}\left[\Gamma\left(\frac{\alpha+1}{\alpha}\right) - \Gamma^2\left(\frac{\alpha+1}{\alpha}\right)\right]$
Normal	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	μ	σ^2
Triangular	$f(x) = \begin{cases} \frac{2(x-a)}{(b-a)(m-a)} \text{ if } a \le x \le m \\ \frac{2(b-x)}{(b-a)(b-m)} \text{ if } m < x \le b \end{cases}$	$\frac{a+b+m}{3}$	$\frac{a^2+b^2+m^2-ab-am-bm}{18}$