Another example: Torsional System

$-\Delta U = \Delta T$

$-\frac{1}{2}k\Theta^2 = \frac{1}{2}J\dot{\Theta}^2$

taking derivative wrt time

$-k\Theta\ddot{\Theta} = J\dot{\Theta}^2$

$J\ddot{\Theta} + k\Theta = 0$

Another Example: U-Tube manometer

$-\Delta U = \Delta T$

$-pV_0h = \frac{1}{2}mx^2$

$-pA\dot{x}g\dot{x} = \frac{1}{2}pA\dot{x}^2$

$-g\dot{x}^2 = \frac{1}{2}L\dot{x}^2$

taking derivative wrt time

$-g2\dot{x}\ddot{x} = \frac{1}{2}2\dot{x}^2$

$\ddot{x} + \frac{2g}{L}x = 0$

Problem 1.65

Pendulum arm of mass $m$
Mass at end of arm $m_T$

$-\Delta U = \Delta T$

$-m_TgL(1-\cos \Theta) - mg\frac{L}{2}(1-\cos \Theta) = \frac{1}{2}m_TL^2\dot{\Theta}^2 + \frac{1}{2}(\frac{1}{2}mL^2)\dot{\Theta}^2$

$(m_T + \frac{m}{2})gL(1-\cos \Theta) + (m_T + \frac{m}{2})\frac{1}{2}L^2\dot{\Theta}^2 = 0$

taking derivative wrt time

$(m_T + \frac{m}{2})gL\sin \Theta \dot{\Theta} + (m_T + \frac{m}{2})L^2 \dot{\Theta} \ddot{\Theta} = 0$

taking $\sin \Theta \approx \Theta$

$\dot{\Theta} + \frac{g}{(m_T + \frac{m}{2})L} \Theta = 0$
Problem 1.67

Given: rack & pinion setup with torsional & lateral spring as shown,

\[ k_1 \quad k_2 \]

\[ \theta \quad \text{m} \]

\[ x \]

Find: 1) "Equation of motion" \( x(t) \) or system differential equation? 2) Natural frequency.

Sol'n:

1) \[ -\Delta U = \Delta T \]

\[ \frac{1}{2} k_1 \theta^2 + \frac{1}{2} k_2 x^2 = \frac{1}{2} I_r \dot{\theta}^2 + \frac{1}{2} m \dot{x}^2 \]

- torsional shaft
- linear spring
- \( K_\text{pinion} \) K.E.

2) \( \theta \) and \( x \) are related \( \Rightarrow x = r \theta \) or \( \theta = \frac{x}{r} \)

- put all \( \theta \) in terms of \( x \)

\[ -\left(\frac{1}{2} k_1 \left(\frac{x}{r}\right)^2 + \frac{1}{2} k_2 x^2\right) = \frac{1}{2} I_r \left(\frac{\ddot{x}}{r}\right)^2 + \frac{1}{2} m \ddot{x}^2 \]

3) Take derivative w.r.t. to time

\[ -\frac{k_1}{2r^2} 2x \ddot{x} - \frac{k_2}{2} 2x \dddot{x} = \frac{I_r}{2r^2} (2\ddot{x}) \dddot{x} + \frac{m}{2} (2\dddot{x}) \dddot{x} \]

4) Cancel \( \ddot{x} \)

\[ -\frac{k_1}{r^2} x - k_2 x = \frac{I_r}{r^2} \dddot{x} + m \dddot{x} \]

5) Put into standard D.E. form

\[ \left(\frac{I_r}{r^2} + m\right) \dddot{x} + \left(\frac{k_1}{r^2} + k_2\right) x = 0 \]

6) Solve for \( \omega_n \), \( x(t) \)

\[ \omega_n = \sqrt{\frac{k_1}{I_r} + k_2}{\sqrt{I_r + m}} \]

\[ x(t) = A \sin (\omega_n t + \phi) \]
Problem 1.73

\[ k = 400 \text{ N\cdotm/\text{rad}} \]

\[ J = 1000 \text{ m}^2 \cdot \text{kg} \]

\[ C = 20 \text{ N\cdotm/\text{rad}} \]

\[ J \ddot{\theta} + c \dot{\theta} + k \theta = 0 \]

\[ \omega_n = \sqrt{\frac{k}{J}} = \sqrt{\frac{400}{1000}} = 0.632 \text{ rad/s} \]

\[ \zeta = \frac{C}{2JK} = \frac{20}{2 \sqrt{400 \cdot 1000}} = 0.0158 \]

\[ \omega_d = \omega_n \sqrt{1 - \zeta^2} = 0.632 \sqrt{1 - 0.0158^2} = 0.632 \text{ rad/s} \]

H. W.: 1.69, 1.70

Trick with 1.70

need to find relationship between \( \theta \) and \( \phi \)
Procedure for Using Energy Method to obtain system Equations

Step 1: \(-\Delta U = \Delta T\)

\[ \Delta U_g = mg(h_2 - h_1) \text{ - gravity} \]
\[ \Delta U_k = \frac{1}{2} k(x_2^2 - x_1^2) \text{ - spring} \]
\[ \Delta U_{kr} = \frac{1}{2} k(\theta_2^2 - \theta_1^2) \text{ - torsional spring} \]
\[ \Delta T_k = \frac{1}{2} m(\dot{x})^2 \text{ - kinetic energy} \]
\[ \Delta T_{kr} = \frac{1}{2} J(\dot{\theta})^2 \text{ - kinetic energy - rotational} \]

- parallel axes theorem

\[ J = J_1 + ml^2 \]

Step 2: Get in terms of one independent variable

Eg: solve for all \( \theta \) in terms of \( x \), or
solve for all \( x \) in terms of \( \theta \)

Step 3: take derivative w.r.t time

Step 4: cancel \( x \) or \( \dot{\theta} \) terms

Step 5: put into nice differential equation form:

- put into same form as \( m\ddot{x} + kx = 0 \)

Step 6: Solve for \( \omega_n \), etc. if required.