Motorcycle Engine Vibration Problem

- A motorcycle engine turns (and vibrates) at **300 rpm** with a harmonic force of **20 N**.
- What is the amplitude of the vibration with respect to the frame (assumed to be stationary) and phase of the response (with respect to the force), if:
  - the mass of the engine is **40 kg**,
  - the stiffness of the mounts is **40 kN/m** and
  - the damping coefficient is **125 Ns/m**?
Damped Spring-Mass System with Forced Vibration

\[ m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t \]

Dividing both sides by \( m \) yields:

\[ \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = f_0 \cos \omega t \]

where:

\[ f_0 = \frac{F_0}{m} \]

Solving the Differential Equation

Assume a particular solution:

\[ x_p(t) = A \cos \omega t + B \sin \omega t \]

so that:

\[ \dot{x}_p(t) = -\omega A \sin \omega t + \omega B \cos \omega t \]

\[ \ddot{x}_p(t) = -\omega^2 (A \cos \omega t + B \sin \omega t) \]

Substituting into the D.E.:

\[ -\omega^2 (A \cos \omega t + B \sin \omega t) + 2\zeta \omega_n \omega (-A \sin \omega t + B \cos \omega t) + \omega_n^2 (A \cos \omega t + B \sin \omega t) = f_0 \cos \omega t \]
Solving the Differential Equation

Separating \( \cos \omega t \) and \( \sin \omega t \) terms yields:

\[
(- \omega^2 A + 2\zeta \omega_n \omega B + \omega_n^2 A - f_0) \cos \omega t \\
+ (- \omega^2 B - 2\zeta \omega_n \omega A + \omega_n^2 B) \sin \omega t = 0
\]

In order to equal zero all the time, both of the constants in front of the \( \cos \omega t \) and \( \sin \omega t \) terms must equal zero. I.e.:

\[
(\omega_n^2 - \omega^2)A + 2\zeta \omega_n \omega B = f_0 \\
(2\zeta \omega_n \omega)A + (\omega_n^2 - \omega^2)B = 0
\]

Solving the Differential Equation

- Solving for \( A \) and \( B \) yields:

\[
A = \frac{(\omega_n^2 - \omega^2)f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta \omega_n \omega)^2} \\
B = \frac{2\zeta \omega_n \omega f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta \omega_n \omega)^2}
\]

- Substituting for \( A \) and \( B \) in \( x_p(t) \) yields:

\[
x_p(t) = \frac{f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta \omega_n \omega)^2} \left( (\omega_n^2 - \omega^2) \cos \omega t + 2\zeta \omega_n \omega \sin \omega t \right)
\]
Solving Differential Equation

- Re-writing in terms of \( X \) and \( \theta \) instead of \( A \) and \( B \) yields:

\[
x_p(t) = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \cos \left( \omega t - \tan^{-1}\left( \frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2} \right) \right)
\]

- Final equation (for **forced vibration** without free vibration):

\[
x_p(t) = X \cos(\omega t - \theta)
\]

where:

\[
X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}
\]

\[
\theta = \tan^{-1}\left( \frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2} \right)
\]

Solving Differential Equation

- If we consider both the **free vibration** and **forced vibration** \( x(t) = x_h(t) + x_p(t) \), solving for amplitude components in terms of \( x_0 \) and \( v_0 \) yields:

\[
x(t) = e^{-\omega_d t} \left\{ \begin{array}{l}
\left[ x_0 - \frac{f_0(\omega_n^2 - \omega^2)}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} \right] \cos \omega_d t \\
+ \left[ \frac{\zeta\omega_d}{\omega_d} \left[ x_0 - \frac{f_0(\omega_n^2 - \omega^2)}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} \right] \right] \sin \omega_d t \\
- \frac{2\zeta\omega_n\omega^2 f_0}{\omega_d \left[ (\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2 \right]} + \frac{v_0}{\omega_d} \\
+ \frac{f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} \left[ (\omega_n^2 - \omega^2) \cos \omega \tau + 2\zeta\omega_n\omega \sin \omega \tau \right]
\end{array} \right. \]

Plotting the “Frequency Response”

Looking at the forced vibration $x_p(t)$, we can plot the amplitude $X$ and phase lag $\theta$ as a function of forcing frequency $\omega$. 

\[ \text{X} \quad \omega \]

\[ \theta \quad \omega \]