Modal Analysis

What we did on the computers last class
Sec. 4.2-4.6

Free Vibration Solution

- In Sec. 4.1 we solved the system differential matrix equation:

- by assuming:

- resulting in the equation:

- which was used to solve for natural frequencies:

- and mode shapes:
Eigenvalues and Eigenvectors

- In Linear Algebra, the Eigenvalue problem is:
  - Given:
    - Matrix \( A \)
    - Matrix equation \( A \mathbf{v} = \lambda \mathbf{v} \)
  - Find solutions for:

- A lot of knowledge is available in mathematics about Eigenvalue problems.

Mapping System Equation to Eigenvalue problem

1. Solve for \( L \) such that \( M = L L^T \)
   - This can be done using a “Cholesky decomposition”
   - This is like solving for the square-root of \( M \).
   - Let’s use \( M^{1/2} \) to refer to \( L \).

2. Solving for inverse of \( L \):
   - \( M^{-1/2} = \text{inverse}(L) \)
3. Introduce new function of time \( q(t) \) such that:
\[
q(t) = Mx(t) \quad \text{or} \quad x(t) = M^{-1/2}q(t)
\]

4. Substituting into system differential equation and pre-multiplying by \( M^{-1/2} \):

5. Assume solution \( q(t) = v e^{j\omega t} \).
Therefore:
Mapping System Equation to Eigenvalue problem

Therefore, to map the system equation to an Eigenvalue problem:

\[ \mathbf{A} = \]

After solving for \( \lambda \) and \( \mathbf{v} \):

\[ \omega_{ni} = \]

\[ \mathbf{u}_i = \]

Mapping Eigenvalue problem to SDOF problem

• The Eigenvalues and Eigenvectors can be used to simplify a MDOF analysis into a several SDOF analyses:
Mapping Eigenvalue problem to SDOF problem

- Define matrix of eigenvectors:
  \[ P = [v_1 \ v_2 \ v_3 \ ... \ v_n] \]

- Matrix of mode shapes is:
  \[ S = [u_1 \ u_2 \ u_3 \ ... \ u_n] = P \]

- Define modal coordinates \( r \) such that:
  \[ q(t) = P \ r(t) \]

Mapping Eigenvalue problem to SDOF problem

- Substituting \( P \ r(t) \) for \( q(t) \) in system equation and pre-multiplying by \( P^T \) yields:
Mapping Eigenvalue problem to SDOF problem

- We therefore have a nice set of SDOF equations:

Modal Analysis

- To solve with initial conditions $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$, use:
  
  $r(0) = $ 
  
  $\dot{r}(0) = $ 

- To get back $\mathbf{x}(t)$ from $r(t)$, use:
  
  $\mathbf{x}(t) = $
Modal Analysis Procedure

1. Calculate $M^{-1/2}$.
2. Calculate $\tilde{K} = M^{-1/2}K M^{-1/2}$, the mass normalized stiffness matrix.
3. Calculate the symmetric eigenvalue problem for $\tilde{K}$ to get $\omega_i^2$ and $v_i$.
4. Normalize $v_i$ and form the matrix $P = [v_1 \ v_2]$.
5. Calculate $S = M^{-1/2} P$ and $S^{-1} = P^T M^{1/2}$.
6. Calculate the modal initial conditions: $r(0) = S^{-1} x_0, \dot{r}(0) = S^{-1} \dot{x}_0$.
7. Substitute the components of $r(0)$ and $\dot{r}(0)$ into equations (4.66) and (4.67) to get the solution in modal coordinate $r(t)$.
8. Multiply $r(t)$ by $S$ to get the solution $x(t) = Sr(t)$.

Note that $S$ is the matrix of mode shapes and $P$ is the matrix of eigenvectors.

Comparing System Representations

<table>
<thead>
<tr>
<th>System D.E.</th>
<th>Original Problem</th>
<th>Eigenvalue Prob.</th>
<th>Modal Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M\ddot{x} + Kx = 0$</td>
<td>$I\ddot{q} + \tilde{K}q = 0$</td>
<td>$I\ddot{r} + \Lambda r = 0$</td>
<td>$r(t) = Ae^{i\omega nt}$</td>
</tr>
<tr>
<td>Form of sol.</td>
<td>$x(t) = ue^{i\omega nt}$</td>
<td>$q(t) = ve^{i\omega nt}$</td>
<td>$A$ is from I.C.</td>
</tr>
<tr>
<td>After sub.</td>
<td>$(K - \omega_n^2 M)u = 0$</td>
<td>$\tilde{K}v = \omega_n^2 v$</td>
<td></td>
</tr>
</tbody>
</table>

$$\tilde{K} = M^{-1/2} K M^{-1/2} \quad \Lambda = \begin{bmatrix} \omega_{n1}^2 & 0 & 0 \\ 0 & \omega_{n2}^2 & 0 \\ 0 & 0 & \omega_{n3}^2 \end{bmatrix}$$

$$u_i = M^{-1/2}v_i \quad r(0) = S^{-1} x(0)$$

$$S = [u_1 \ u_2 \ u_3] \quad x(t) = Sr(t)$$
"Nodes" of a Mode

- These are places where the mode shape is zero.

- Not a good place to mount a sensor or actuator for body motion.
- Good place to mount devices that shouldn’t receive or transmit vibrations at the given natural frequency.

Rigid-Body Modes

- Appear as natural frequencies with value of zero

- Require special treatment when evaluating motion from initial conditions (see p. 314 in text)
Viscous Damping

- It is relatively difficult to model individual dampers in a Modal Analysis.
- Some “tricks” are available:
  - “Modal damping” (apply damping $\zeta_i$ to system equation for each mode in modal coordinates $r(t)$)
  - “Proportional damping” ($C = \alpha M + \beta K$, with $\alpha$ and $\beta$ chosen freely)

$$\zeta_i = \frac{\alpha}{2\omega_i} + \frac{\beta\omega_i}{2} \quad i = 1, 2, \ldots, n$$

Forced Response

Forces can be mapped to modal equations

1. $$M\ddot{x} + C\dot{x} + Kx = BF(t)$$
   
   $$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad F(t) = \begin{bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \\ F_4(t) \end{bmatrix}$$

2. $$I\ddot{q}(t) + \tilde{C}\dot{q}(t) + \tilde{K}q(t) = M^{-1/2}BF(t)$$
   $$\tilde{C} = M^{-1/2}CM^{-1/2}.$$ 

3. $$\ddot{r}(t) + \text{diag}[2\zeta_i\omega_i]\dot{r}(t) + \Lambda r(t) = P^TM^{-1/2}BF(t)$$
   $$\ddot{r}_i(t) + 2\zeta_i\omega_i\dot{r}_i(t) + \omega_i^2r_i(t) = f_i(t)$$