Some matrices and a scalar:

\[ A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, \quad C = \begin{bmatrix} 9 \\ 10 \end{bmatrix}, \quad D = \begin{bmatrix} 11 \\ 12 \end{bmatrix}, \quad E = 13. \]

Matrix operations:

- **Adding**
  \[ A + B = B + A = \begin{bmatrix} 1 + 5 & 2 + 6 \\ 3 + 7 & 4 + 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix} \]

- **Subtracting**
  \[ B - A = \begin{bmatrix} 5 - 1 & 6 - 2 \\ 7 - 3 & 8 - 4 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \]

- **Multiplying**
  \[ AE = EA = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 13 \\ 26 \end{bmatrix} = \begin{bmatrix} 1 \times 13 & 2 \times 13 \\ 3 \times 13 & 4 \times 13 \end{bmatrix} = \begin{bmatrix} 13 & 26 \\ 39 & 52 \end{bmatrix} \]
  \[ AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 \times 5 + 2 \times 7 & 1 \times 6 + 2 \times 8 \\ 3 \times 5 + 4 \times 7 & 3 \times 6 + 4 \times 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} \]

The product of matrices is not commutative: \[ BA \neq AB \]

\[ AC = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 9 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \times 9 + 2 \times 10 \\ 3 \times 9 + 4 \times 10 \end{bmatrix} = \begin{bmatrix} 29 \\ 67 \end{bmatrix} \]

\[ CA \neq AC \]

\[ DA = (11 \quad 12) \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = (11 \times 1 + 12 \times 3 \quad 11 \times 2 + 12 \times 4) = \begin{bmatrix} 47 \\ 70 \end{bmatrix} \]

- **Transpose**
  \[ A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix} \]
  \[ C^T = \begin{bmatrix} 9 \\ 10 \end{bmatrix}^T = [9 \quad 10] \]
  \[ D^T = \begin{bmatrix} 11 \\ 12 \end{bmatrix}^T = \begin{bmatrix} 11 \\ 12 \end{bmatrix} \]

The “Identity Matrix” has 1s on the diagonals and 0s everywhere else.

- **Inverse**
  \[ AA^{-1} = A^{-1}A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

Solving for \( A^{-1} \) may be difficult. In fact, \( A^{-1} \) may not even exist.
“Solving” usually refers to solving for \( \mathbf{D} \) in the equation:

\[
\mathbf{K} \mathbf{D} = \mathbf{R}
\]  

(1)

where \( \mathbf{K} \) and \( \mathbf{R} \) are known. \( \mathbf{D} \) and \( \mathbf{R} \) are usually column vectors, while \( \mathbf{K} \) is a square matrix.

It is possible to solve for the matrix \( \mathbf{K}^{-1} \) and pre-multiply both sides of equation (1) so that:

\[
\mathbf{K}^{-1} \mathbf{K} \mathbf{D} = \mathbf{K}^{-1} \mathbf{R}
\]

\[
\mathbf{D} = \mathbf{K}^{-1} \mathbf{R}
\]

however this is more work than necessary, since it is possible to solve for \( \mathbf{D} \) directly without solving for \( \mathbf{K}^{-1} \) first.

Solving algorithms can be:

- Iterative; starting with a guess at the solution, repeated approximations are applied until numerical convergence is achieved, or
- Direct; repeated substitutions are employed until all matrix elements have been considered (e.g., Gaussian Elimination).

In a direct solver, the solution time is roughly proportional to \( nb^2 \), where \( n \) is the number of rows/columns in \( \mathbf{K} \) and \( b \) is the bandwidth. The bandwidth is the maximum number of columns in a row from the first non-zero column to the last non-zero column, as shown below. A “sparse” matrix is a matrix that has mostly zeros. Which matrix elements are nonzero is determined by the node numbering and connections, however in a sparse matrix, the nodes can usually be re-numbered to give a small bandwidth, as shown below. In modern FEA software the nodes are renumbered automatically to provide for the most efficient solution.