Beam Elements
Bending of Beams – Definition of Problem

Beam with a straight axis \((x\text{-axis})\), with varying cross-section \((A(x))\), loaded transversely \((q(x) \text{ in } y \text{ direction})\) with transverse deflections \((v(x) \text{ in } y \text{ direction})\).
Bending of Beams – Definition of Problem

\[ \varepsilon(x) = -\frac{y}{\rho} \]

\[ \frac{1}{\rho} \cong \frac{d^2v}{dx^2} \]

\[ \sigma = E \varepsilon = -Ey \frac{d^2v}{dx^2} \]

\[ \int_{A} \sigma y dA = \int_{A} -Ey^{2} \frac{d^2v}{dx^2} dA \]

MAE 456 Finite Element Analysis
Bending of Beams – Definition of Problem

\[ \int_A \sigma_y dA = \int_A - Ey^2 \frac{d^2 v}{dx^2} dA \]

\[ M = -EI \frac{d^2 v}{dx^2} \]

\[ V = -\frac{dM}{dx} = \frac{d}{dx} EI \frac{d^2 v}{dx^2} \]

\[ q(x) = \frac{dV}{dx} = \frac{d^2}{dx^2} EI \frac{d^2 v}{dx^2} \]

Continued from previous slide.

Resulting Field Equation
### Bending of Beams – Definition of Problem

**Summary**

Field equation:  
\[ EI \frac{d^4v}{dx^4} = q(x) \]

Slope:  
\[ \theta(x) = \frac{dv}{dx} \]

Bending Moment:  
\[ M(x) = EI \frac{d^2v}{dx^2} \]

Shear Force:  
\[ V(x) = -EI \frac{d^3v}{dx^3} \]
Bending of Beams – Definition of Problem

Boundary Conditions: Four boundary conditions are necessary to solve a bending problem. Boundary conditions can be:

- **deflection:** \( v(0), v(L) \)
- **slope:** \( \theta(0), \theta(L) \)
- **bending Moment:** \( M(0), M(L) \)
- **shear force:** \( V(0), V(L) \)
Beam Element – Shape Functions

- Recall that shape functions are used to interpolate displacements.
Beam Element – Shape Functions

\[ v = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_z^1 \\ v_2 \\ \theta_z^2 \end{bmatrix} = \mathbf{N} \mathbf{d} \]

• There are two degrees of freedom (displacements) at each node: \( v \) and \( \theta_z \).
• Each shape function corresponds to one of the displacements being equal to ‘one’ and all the other displacements equal to ‘zero’.
• Note that everything we do in this course assumes that the displacements are small.
Beam Element – Shape Functions

\[
\sigma = E\varepsilon = -Ey\frac{d^2v}{dx^2} = -Ey\left[\frac{d^2N}{dx^2}\right]d = -EyBd
\]

\[
B = \begin{bmatrix}
-\frac{6}{L^2} + \frac{12x}{L^3} & -\frac{4}{L} + \frac{6x}{L^2} & \frac{6}{L^2} - \frac{12x}{L^3} & -\frac{2}{L} + \frac{6x}{L^2}
\end{bmatrix}
\]

- Recall that the B row vector is used to interpolate stresses & strains.
Beam Element – Formal Derivation

- The formal beam element stiffness matrix derivation is much the same as the bar element stiffness matrix derivation. From the minimization of potential energy, we get the formula:

$$k = \int_0^L B^T EI B \, dx$$

- As with the bar element, the strain energy of the element is given by $$\frac{1}{2} d^T k d.$$
Beam Element – Formal Derivation

The result is:

\[ k = \frac{EI}{L^3} \begin{bmatrix}
12 & 6L & -12 & 6L \\
6L & 4L^2 & -6L & 2L^2 \\
-12 & -6L & 12 & -6L \\
6L & 2L^2 & -6L & 4L^2
\end{bmatrix} \]

which operates on \( d = \begin{bmatrix} v_1, \theta_{z1}, v_2, \theta_{z2} \end{bmatrix}^T \).
Beam Element – Formal Derivation

• The moment along the element is given by:

\[ M = EI \frac{d^2v}{dx^2} = EI \, Bd \]

• The stress is given by:

\[ \sigma_x = -\frac{My}{I} = -yEBd \]
Beam Element w/Axial Stiffness

- If we combine the bar and beam stiffness matrices, we get a general beam stiffness matrix with axial stiffness.

\[
k = \begin{bmatrix}
AE/L & 0 & 0 & -AE/L & 0 & 0 \\
0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\
0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\
-\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\
0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\
0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L
\end{bmatrix}
\]
Uniformly Distributed Loads

Laterally:

Element

Structure

Laterally:
Equivalent Loadings

TABLE 4.2 Equivalent nodal loading of beams

<table>
<thead>
<tr>
<th>Loading</th>
<th>Equivalent Nodal Loading</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w$</td>
<td>$\frac{wL}{2}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{wL^2}{12}$</td>
</tr>
<tr>
<td>$wL$</td>
<td>$\frac{3wL}{20}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{wL^2}{30}$</td>
</tr>
<tr>
<td>$P$</td>
<td>$\frac{P}{2}$</td>
</tr>
<tr>
<td>$\frac{L}{2}$</td>
<td>$\frac{PL}{8}$</td>
</tr>
<tr>
<td>$\frac{L}{2}$</td>
<td>$\frac{PL}{8}$</td>
</tr>
</tbody>
</table>

$M = \frac{PL}{8}$

$M = \frac{PL}{8}$
Orientating Element in 3-D Space

- Transformation matrices are used to transform the equations in the element coordinate system to the global coordinate system, as was shown for the bar element.