Planar Problems

Structural problems that can be described in 2 dimensions.
Stresses on an Arbitrary Plane

• Example:

• Using Mohr’s Circle
Stresses on an Arbitrary Plane

\[
\begin{align*}
\sigma_n &= \sigma_x n_x^2 + 2\tau_{xy} n_x n_y + \sigma_y n_y^2 \\
\tau_n &= -\sigma_x n_x n_y + (n_x^2 - n_y^2)\tau_{xy} + \sigma_y n_x n_y
\end{align*}
\]

or

\[
\begin{align*}
\sigma_n &= \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos(2\phi) + \tau_{xy} \sin(2\phi) \\
\tau_n &= -\frac{1}{2}(\sigma_x - \sigma_y)\sin(2\phi) + \tau_{xy} \cos(2\phi)
\end{align*}
\]

\[
n = n_x \hat{i} + n_y \hat{j} = \cos \phi \hat{i} + \sin \phi \hat{j}
\]
Principal Stresses

- Principal stresses occur on planes that have zero shear stress.
- For planar problems, this condition is:

\[-\sigma_x n_x n_y + \left(n_x^2 - n_y^2\right)\tau_{xy} + \sigma_y n_x n_y = 0\]

\[\tan(2\phi) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}\]

\[n_x = \cos(\phi)\]

\[n_y = \sin(\phi)\]
Principal Stresses

• Maximum normal stress at a point

\[ \sigma_1 = \]

• Minimum normal stress at a point

\[ \sigma_2 = \]

• Maximum shear stress at a point

\[ \tau_{\text{max}} = \]
Theories of Failure

• Ductile Material
  – Maximum Shear Stress Theory
  – Maximum Distortion Energy Theory
  • von Mises Stress
    \[
    \sigma_e = \left( \frac{1}{\sqrt{2}} \right) \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2}
    \]

• Brittle Material
  – Maximum Normal Stress Theory
  – Coulomb-Mohr Theory
Theories of Failure

Image from J.A. Collins, “Failure of Materials in Mechanical Design,” Wiley
Theories of Failure

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Stress-Strain Relations

- The relationship between stress and strain is given by: \( \sigma = E \varepsilon + \sigma_0 \)

where:

\[
\sigma = \begin{cases} 
\sigma_x \\ \sigma_y \\ \tau_{xy} 
\end{cases} \quad \varepsilon = \begin{cases} 
\varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} 
\end{cases}
\]

\( \sigma_0 \) is a pre-stress, where \( \sigma_0 = -E \varepsilon_0 \) and \( \varepsilon_0 \) is a pre-strain (usually zero; but would be non-zero for thermal strain, strain hardening, etc.).
Stress-Strain Relations

• The matrix $E$ depends on what you assume is happening in the $z$ direction.

• Two situations can be assumed:
  – plane stress ($\sigma_z = 0$), or
  – plane strain ($\varepsilon_z = 0$).
Plane Stress Problem

- A plane stress problem is one in which the cross-section is **free to move** in the z direction.
- The thickness in the z direction is normally small compared to the profile.
- \( \sigma_z = \tau_{yz} = \tau_{zx} = 0 \)

\[
E = \frac{E}{1-v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & (1-v)/2 \end{bmatrix}
\]

for plane stress
Plane Stress Problem

• The inverse relationship (between strain and stress for plane stress) is given by:

\[ \varepsilon = \mathbf{E}^{-1} \sigma + \varepsilon_0 \]

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} =
\begin{bmatrix}
1/E & -\nu/E & 0 \\
-\nu/E & 1/E & 0 \\
0 & 0 & 1/G
\end{bmatrix}
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} +
\begin{bmatrix}
\varepsilon_{x0} \\
\varepsilon_{y0} \\
\gamma_{xy0}
\end{bmatrix}
\]

\[ G = \frac{E}{2(1+\nu)} \]
Plane Strain Problem

- A plane strain problem is one in which the cross-section is **restrained from moving** in the z direction.
- The thickness in the z direction is normally very large compared to the cross section.

- $\varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0$
- $\gamma_{yz} = \gamma_{zx} = 0$ but $\sigma_z = \nu(\sigma_x + \sigma_y)$

\[
E = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix}
(1-\nu) & \nu & 0 \\
\nu & (1-\nu) & 0 \\
0 & 0 & (1-2\nu)/2
\end{bmatrix}
\text{ for plane strain}
\]
Plane Strain Problem

- The inverse relationship (between strain and stress for plane strain) is given by:

\[ \varepsilon = \mathbf{E}^{-1} \sigma + \varepsilon_0 \]

\[
\mathbf{E}^{-1} = \begin{bmatrix}
\frac{1-\nu^2}{E} & \frac{-\nu-\nu^2}{E} & 0 \\
\frac{-\nu-\nu^2}{E} & \frac{1-\nu^2}{E} & 0 \\
0 & 0 & \frac{1}{G}
\end{bmatrix}
\]

\[ G = \frac{E}{2(1+\nu)} \]
Strain-Displacement Relations

\[ \varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \]

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 \\
0 & \frac{\partial}{\partial y} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x}
\end{bmatrix}
\begin{bmatrix}
u \\
v
\end{bmatrix}
\text{or } \varepsilon = \partial u\]