Elements of Revolution

Axisymmetric problems!

Axisymmetric Elements

- In axisymmetric problems,
  - the geometry is axisymmetric and
  - the loads and supports are axisymmetric, (though there are some tricks to create non-axisymmetric loads.

- Meshing is similar to plane problems, however:
  - each node represents a circle and
  - each element represents the cross section of an annulus.
Axisymmetric Elements

• Stress-strain relations

\[
\begin{bmatrix}
\sigma_r \\
\sigma_\theta \\
\sigma_z \\
\tau_{\theta z}
\end{bmatrix} =
\begin{bmatrix}
(1 - \nu)c & \nu c & \nu c & 0 \\
(1 - \nu)c & \nu c & 0 & 0 \\
(1 - \nu)c & 0 & \nu c & 0 \\
\text{symmetric} & & & G
\end{bmatrix}
\begin{bmatrix}
\varepsilon_r \\
\varepsilon_\theta \\
\varepsilon_z \\
\gamma_{\theta z}
\end{bmatrix}
\]

• Strain-displacement relations

\[
\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\theta = \frac{u}{r}
\]

\[
\varepsilon_z = \frac{\partial w}{\partial z}, \quad \gamma_{yz} = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}
\]

Initial arc = \(a_0 = r \, d\theta\)
Displaced arc = \(a = (r + u) \, d\theta\)
By definition, \(\varepsilon_\theta = \frac{a - a_0}{a_0}\)
Hence \(\varepsilon_\theta = \frac{u}{r}\)
Axisymmetric Elements

- Displacement interpolation

\[
\begin{bmatrix}
u \\
w
\end{bmatrix} = \begin{bmatrix}
N_1 & 0 & N_2 & 0 & \cdots \\
0 & N_1 & 0 & N_2 & \cdots
\end{bmatrix} \begin{bmatrix}
u_1 \\
w_1 \\
u_2 \\
w_2 \\
\vdots
\end{bmatrix}
\]

- General formula for \( k \) (axisymmetric stress field)

\[
\varepsilon = B \cdot d
\]

\[
\begin{align*}
\varepsilon & = B \cdot d \\
\begin{array}{c}
4 \times 1 \\
4 \times 2n \\
2n \times 1
\end{array}
\end{align*}
\]

\[
\begin{align*}
k & = \int \int \int B^T \cdot E \cdot B \cdot r \ dr \ d\theta \ dz \\
\begin{array}{c}
2n \times 2n \\
4 \times 4
\end{array}
\end{align*}
\]
Axisymmetric Elements

• Except for circumferential strain of $\varepsilon_\theta = u/r$, these elements are similar to plane elements.
• Three node triangle is similar to constant strain triangle:

\[
\begin{align*}
  u &= \beta_1 + \beta_2 r + \beta_3 z \\
  w &= \beta_4 + \beta_5 r + \beta_6 z
\end{align*}
\]

• However, $\varepsilon_\theta$ is a function of $r$ and $z$ (it is not constant strain).

\[
\begin{align*}
  \varepsilon_r &= \beta_2 \\
  \varepsilon_\theta &= \beta_1 \frac{1}{r} + \beta_2 + \beta_3 \frac{z}{r} \\
  \varepsilon_z &= \beta_6 \\
  \gamma_{zr} &= \beta_5 + \beta_3
\end{align*}
\]
Axisymmetric Elements

- Rigid body motion is possible in the z direction, but not in the r direction; nor is rigid body rotation (rolling) possible.

- Note that:

- Isoparametrically:
Nonaxisymmetric Loads

- We can use Fourier Series to represent loads as a function of $\theta$!

$$q = \sum_{n=0}^{\infty} q_{cn} \cos n\theta + \sum_{n=1}^{\infty} q_{sn} \sin n\theta$$
Nonaxisymmetric Loads

• We can also use Fourier Series to represent the displacements as functions of $\theta$!

\[
\begin{align*}
  u &= \sum_{n=0}^{\infty} u_{cn} \cos n\theta + \sum_{n=1}^{\infty} u_{sn} \sin n\theta \\
  v &= \sum_{n=1}^{\infty} v_{cn} \sin n\theta + \sum_{n=0}^{\infty} v_{sn} \cos n\theta \\
  w &= \sum_{n=0}^{\infty} w_{cn} \cos n\theta + \sum_{n=1}^{\infty} w_{sn} \sin n\theta
\end{align*}
\]
Nonaxisymmetric Loads

• Each harmonic can be solved independently.

\[ u_{cn} = \sum N_i u_{cni} \]
\[ v_{cn} = \sum N_i v_{cni} \]
\[ w_{cn} = \sum N_i w_{cni} \]

\[ K_n D_n = R_n \]