Continuous-Time Signals

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EE 327

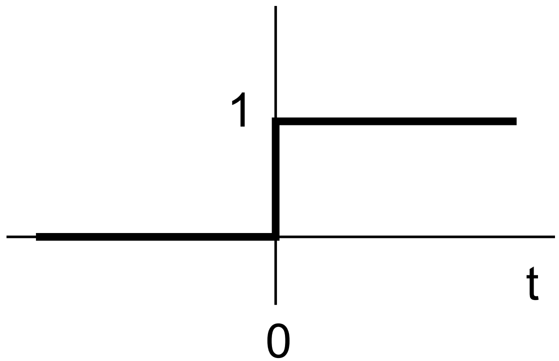
Continuous-Time Signals

- Continuous-Time Signals
  - Time is a continuous variable
  - The signal itself need not be continuous

- We will look at several common continuous-time signals and also operations that may be performed on them

Unit Step Function → u(t)

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



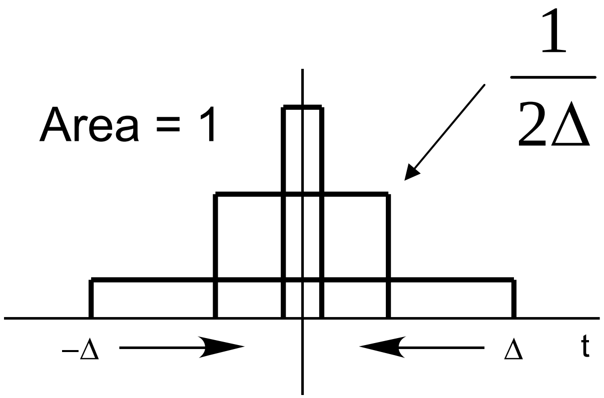
- Used to characterize systems
- We will use u(t) to illustrate the properties of continuous-time signals

Unit Impulse/Delta Function → δ(t)

- Used for complete characterization of systems
- Response of a system to δ(t) allows us to know the response to all signals
- Can approximate any arbitrary waveform/signal
- Not a function
- It is a distribution
- Difficult to make in reality, but it can be approximated

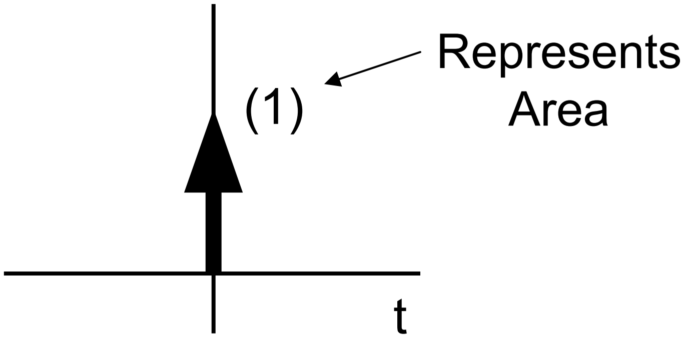
Unit Impulse/Delta Function → δ(t)

Let $\delta_\Delta(t) = \begin{cases} \dfrac{1}{2\Delta} & -\Delta < t < \Delta \\ 0 & otherwise \end{cases}$



$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

- Infinitely narrow
- Infinitely tall
- Always has area = 1



$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ undefined & t = 0 \end{cases}$$

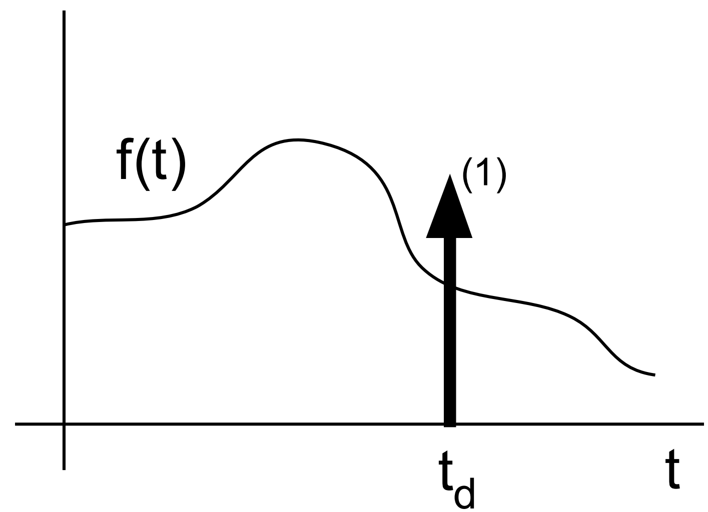
and

$$\int_{-\infty}^{\infty} \delta(\tau) d\tau = 1$$ (Area = 1)

Properties of δ(t)

<u>Sifting Property</u>

- Samples an arbitrary waveform at a given time instance
- Given an arbitrary function f(t) and an impulse $\delta(t - t_d)$, we can find the instantaneous value of $f(t_d)$
  - Multiply the two signals together
  - Integrate because δ(t) is a distribution



Dummy variable of integration

$$\int_{-\infty}^{\infty} f(\tau)\delta(\tau - t_d)\,d\tau =$$

$$= \int_{-\infty}^{\infty} f(t_d)\delta(\tau - t_d)\,d\tau = f(t_d)$$

Because δ(t) = 0 for all values but t_d

Can use this property to sample a CT signal to the DT domain

Sifting Property

For a delayed version of f(t) → f(t – t_1), the sifting property gives us a delayed version of the instantaneous value

$$\int_{-\infty}^{\infty} f(\tau - t_1)\delta(\tau - t_d)d\tau = \int_{-\infty}^{\infty} f(t_d - t_1)\delta(\tau - t_d)d\tau = f(t_d - t_1)$$

ex. Find the instantaneous value (sample) of x(t) = $\sin^2$(t/b) at time t=a

$$\int_{-\infty}^{\infty} \sin^2\left(\frac{\tau}{b}\right)\delta(\tau - a)d\tau = \int_{-\infty}^{\infty} \sin^2\left(\frac{a}{b}\right)\delta(\tau - a)d\tau =$$

$$= \sin^2\left(\frac{a}{b}\right)\int_{-\infty}^{\infty} \delta(\tau - a)d\tau = \sin^2\left(\frac{a}{b}\right)$$

Properties of δ(t)

Relationship to the Step Function

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$$

Area of δ(t) always equals 1

$$\delta(t) = \frac{du(t)}{dt}$$

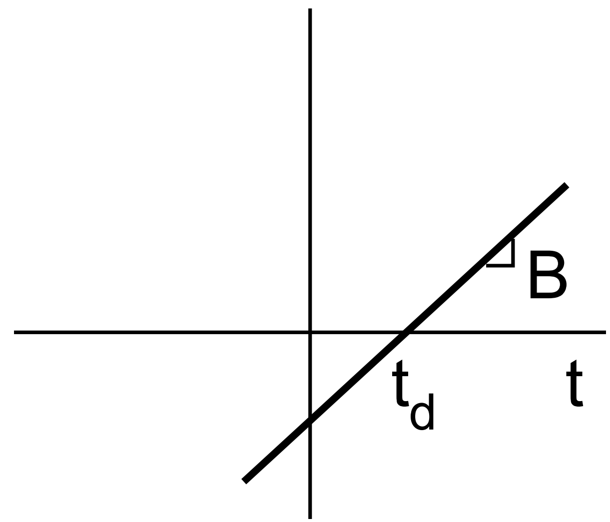
Other Properties

$$\delta(t) = \delta(-t)$$

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

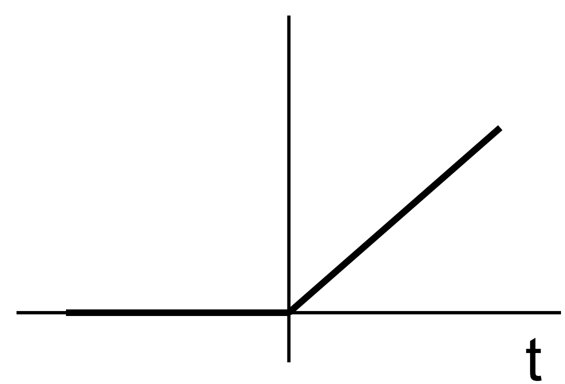
Ramp Functions

- x(t) = t
- Shifted ramp = B(t – t_d)



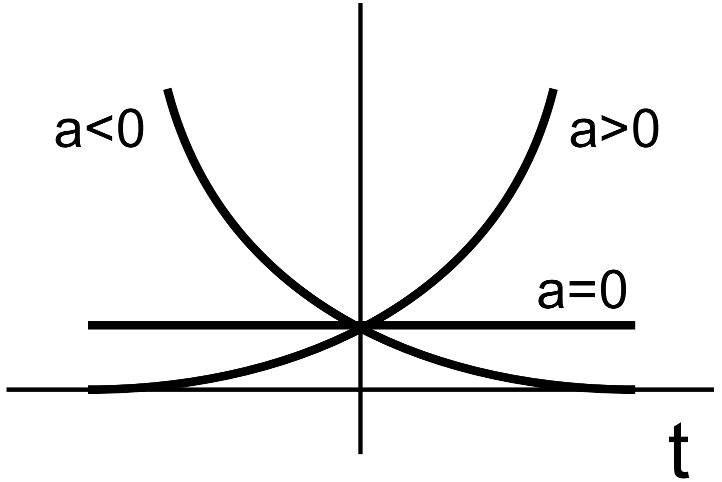
- Unit ramp function, r(t)
  - “Starts” at a given time

$$r(t) = tu(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$



Exponential Signals

- x(t) = Ce^{at}
- if "C" and "a" are real



- If "a" is imaginary

Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Inverse Euler's Formula

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

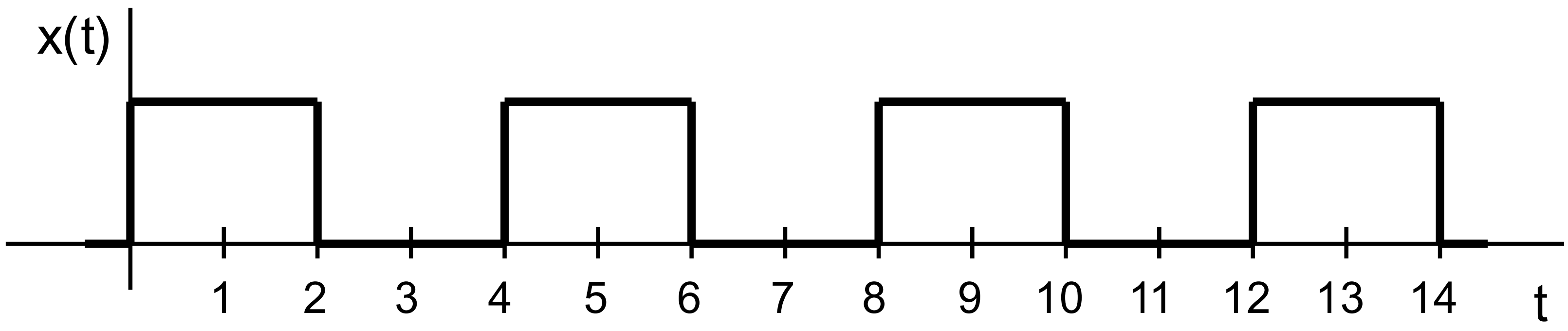
$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Periodic Signals

- Signal that repeats itself every T seconds
  - T=period of the signal
- A signal is periodic if
  - $x(t) = x(t + T)$, where $T>0$ for all "t"
  - Therefore, replace "t" with "t+T"
  - $x(t + T) = x(t + 2T)$
  - Also, $x(t) = x(t + nT)$, n = integer
- Fundamental period = minimum T that satisfies $x(t) = x(t + T)$
  - T_0
  - $f_0 = 1/T_0$
  - $\omega_0 = 2\pi f_0 = 2\pi/T_0$

Periodic Signals

ex. Is x(t) periodic? If so, find the fundamental period of x(t)



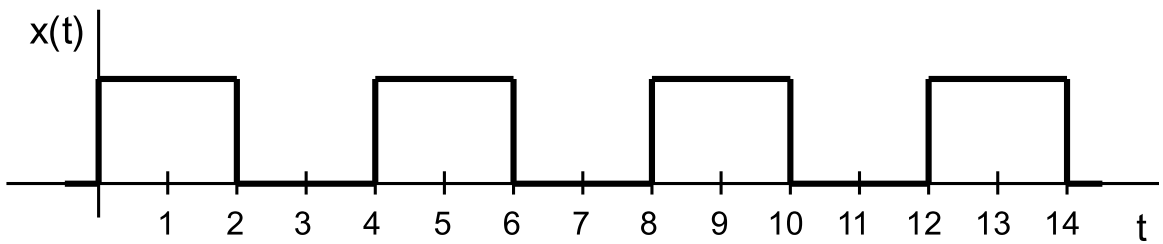
x(t) is periodic with fundamental period $T_0 = 4$

because $x(t) = x(t + 4)$ for all values of "t"

Periodic Signals

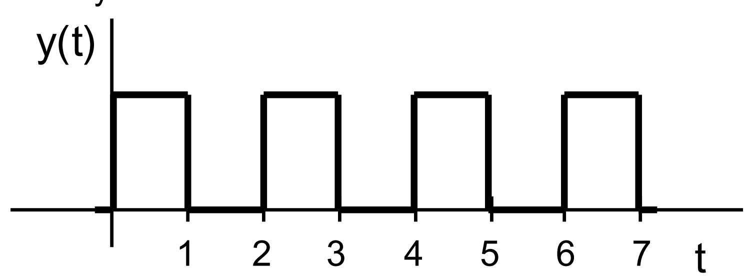
Time scaling applied to periodic signals

- Let y(t) = x(at)
- y(t) has period = T/|a|



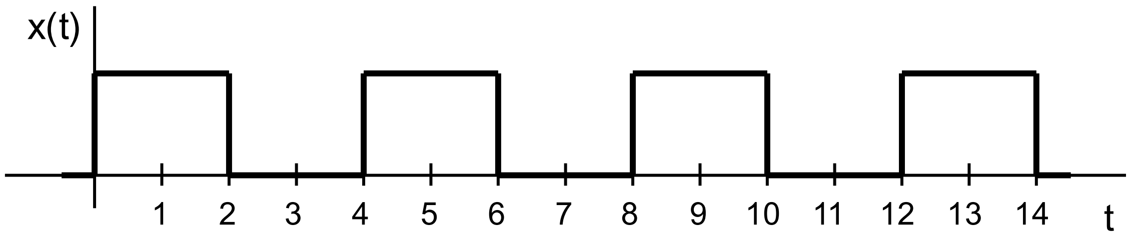
ex. Let y(t) = x(2t), sketch y(t) and find the fundamental period of y(t)

The period of y(t), $T_y = T/|2| = 4/|2| = 2$



Periodic Signals

ex. What is an equation for x(t)



Remember, T_x = 4, so everything repeats every 4 seconds
Therefore, look at only one period

$$x(t) = u(t) - u(t-2) = u(t-4) - u(t-6) = u(t-8) - u(t-10)$$

$$\therefore x(t) = \sum_{m=-\infty}^{\infty} \left[u(t-4m) - u(t-4m-2) \right]$$

$$= \sum_{m=-\infty}^{\infty} \left[u(t-4m) - u(t-(2+4m)) \right]$$

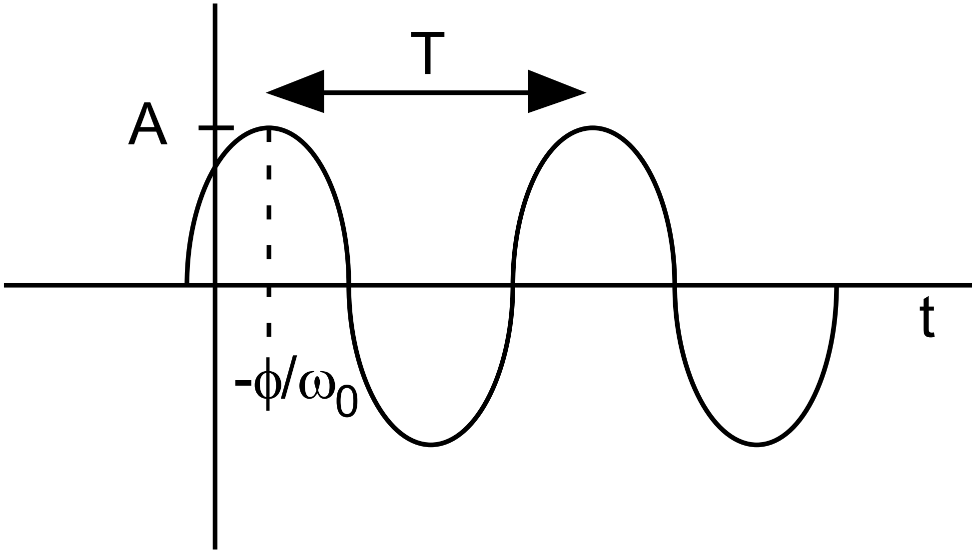
Summation with delay given by the period

Sinusoidal Waveforms

$$x(t) = A\cos(\omega_0 t + \phi) = A\sin\left(\omega_0 t + \phi + \frac{\pi}{2}\right)$$

A = Amplitude
ω_0 = Radian Frequency
φ = Phase Delay

$$\omega_0 = 2\pi f = \frac{2\pi}{T}$$



Sinusoidal Waveforms

Time Shift

Let $x(t) = A\cos(\omega_0 t)$

$$x(t - t_d) = A\cos(\omega_0 (t - t_d)) = A\cos(\omega_0 t + \phi)$$
$$= A\cos(\omega_0 t - \omega_0 t_d) = A\cos(\omega_0 t + \phi)$$

$$\therefore \phi = -\omega_0 t_d$$

$$t_d = \frac{-\phi}{\omega_0} = \frac{-\phi}{2\pi f_0}$$

$$\Rightarrow \phi = -2\pi f_0 t_d = -2\pi \frac{t_d}{T}$$

$\boxed{\text{Time Shift = Time Delay}}$

Sinusoidal Waveforms

ex. What is the time delay of x(t)?

$$x(t) = 10\cos(30\pi t - 0.2\pi)$$

<u>Answer</u>

$$t_d = \frac{-\phi}{\omega_0} = \frac{0.2\pi}{30\pi} = \frac{1}{150}\sec$$

Operations of CT Signals

1. Time Reversal $y(t) = x(-t)$
2. Time Shifting $y(t) = x(t-t_d)$
3. Amplitude Scaling $y(t) = Bx(t)$
4. Addition $y(t) = x_1(t) + x_2(t)$
5. Multiplication $y(t) = x_1(t)x_2(t)$
6. Time Scaling $y(t) = x(at)$