Summation - Tricks of the trade

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1 Factoids

Lemma 1.1 Linearity of sums
\[ \sum_{k=1}^{n}(c.a_k + b_k) = c.\sum_{k=1}^{n}a_k + \sum_{k=1}^{n}b_k \] (1)

Lemma 1.2 Given the arithmetic progression, of the form
\[ a_1, a_1 + d, a_1 + 2d, \ldots, \]
the \( n \)th term in the sequence is given by
\[ T_n = a_1 + (n - 1)d \] (2)
and the sum of the first \( n \) terms is given by:
\[ S_n = \frac{n}{2}(a_1 + T_n) \] (3)

Lemma 1.3 Given a geometric series of the form
\[ 1, x, x^2, \ldots, \]
the \( n \)th term in the sequence is given by
\[ T_n = x^{n-1} \] (4)
and the sum of the first \( n \) terms is given by:
\[ S_n = \frac{1 - x^n}{1 - x} \] (5)

If \( x < 1 \), we can rewrite \( S_n \) as:
\[ S_n = \frac{1}{1 - x} \] (6)

Remark 1.1 These formulae are slightly different from the ones in the text. Ensure that you understand the differences.

Remark 1.2 Prove all the above formulae using mathematical induction.
2 Techniques

The following techniques are used to derive upper and lower bounds on series:

1. Telescoping - If the summand can be expressed as the difference between adjacent terms of the corresponding sequence, then telescoping can be used. Clearly, in sequence

\[ a_0, a_1, \ldots, a_n \]

\[ \sum_{k=1}^{n} a_k - a_{k-1} = a_n - a_0 \]

Similarly,

\[ \sum_{k=0}^{n-1} a_k - a_{k+1} = a_0 - a_n \]

Let us say that we want

\[ \sum_{k=1}^{n-1} \frac{1}{k(k+1)} \]

Observe that \( \frac{1}{k(k+1)} \) can be rewritten as:

\[ \frac{1}{k} - \frac{1}{k+1} \]

Applying the telescoping formula,

\[ \sum_{k=1}^{n-1} \frac{1}{k(k+1)} = 1 - \frac{1}{n} \]

Remark: 2.1 Make sure that you understand every step!

2. Mathematical Induction - The hammer! It can be used almost everywhere. For deriving bounds, it is necessary to guess a value and then validate the guess. (Unfortunately, there will not be a samaritan lurking in the exam/quiz to give you the answer. You will have to make an intelligent guess and then check it.) You do not have to guess the exact function e.g. \( \frac{n(n+1)}{2} \); it suffices to guess the form i.e. \( O(n^2) \).

Consider the geometric series \( \sum_{k=0}^{n} 3^k \). You can apply the formula, which gives \( \frac{3^{n+1} - 1}{2} \) i.e. \( \frac{3}{2} \cdot 3^n - \frac{1}{2} \) i.e. \( O(3^n) \). Or you could guess the form as \( O(3^n) \). This means that we have to prove

\[ \sum_{k=0}^{n} 3^k \leq c \cdot 3^n \]

for some \( c > 0 \).

Base confirmation: Putting \( n = 0 \), gives \( 1 \leq c \) or \( c \geq 1 \). Inductive step: Assuming that the proposition holds for \( n = p \), we get

\[ \sum_{k=0}^{p} 3^k \leq c \cdot 3^p \]

for some \( c > 0 \). Now,

\[ \sum_{k=0}^{p+1} 3^k = \sum_{k=0}^{p} 3^k + 3^{p+1} \]

\[ \Rightarrow \sum_{k=0}^{p+1} 3^k \leq c \cdot 3^p + 3^{p+1} \]
by the inductive hypothesis

\[ \sum_{k=0}^{p+1} 3^k \leq \left( \frac{1}{3} + \frac{1}{c} \right)c.3^{p+1} \]

which holds as long as \( \left( \frac{1}{3} + \frac{1}{c} \right) \leq 1 \) i.e. \( c \geq \frac{3}{2} \). Thus we have proved our case i.e.

\[ \sum_{k=0}^{n} 3^k \leq c.3^n \]

for some \( c \geq \frac{3}{2} \) or

\[ \sum_{k=0}^{n} 3^k = O(3^n) \]

3. Term bounding - Yet another idea is to bound the individual terms of the series, e.g.

\[ \sum_{k=1}^{n} k \leq \sum_{k=1}^{n} n = n^2 \]

For a geometric series, \( \sum_{k=0}^{n} a_k \), if \( \frac{a_{k+1}}{a_k} \leq r < 1 \), where \( r \) is a constant, then \( a_k \leq a_0 . r^k \). Hence,

\[ \sum_{k=0}^{n} a_k \leq \sum_{k=0}^{\infty} a_0 . r^k \]

\[ = a_0 . \sum_{k=0}^{\infty} r^k \]

\[ = a_0 . \frac{1}{1-r} \]

**Remark: 2.2** Use the term bounding idea to derive an upper bound for \( \sum_{k=1}^{\infty} \frac{k}{3^k} \). (Hint: Given in textbook)

4. Sum splitting - This technique is particularly useful for deriving lower bounds e.g.

\[ \sum_{k=1}^{n} k = \sum_{k=1}^{n} k + \sum_{k=\frac{n}{2}+1}^{n} k \]

\[ \Rightarrow 0 + \sum_{k=\frac{n}{2}+1}^{n} \frac{n}{2} \]

\[ \Rightarrow \geq \left( \frac{n}{2} \right)^2 \]

5. Integration - This is a versatile technique used in deriving upper and lower bounds when the function (summand) is not amenable to the tricks above. For a monotonically increasing function, we have

\[ \int_{m-1}^{n} f(x) \, dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m}^{n+1} f(x) \, dx \]

For a monotonically decreasing function, we have

\[ \int_{m-1}^{n} f(x) \, dx \geq \sum_{k=m}^{n} f(k) \geq \int_{m}^{n+1} f(x) \, dx \]
Take $f(k) = \frac{1}{k}$, a monotonically decreasing function. Hence,

$$
\sum_{k=1}^{n} \frac{1}{k} \geq \int_{1}^{n+1} \frac{dx}{x} = \ln(n + 1)
$$

Likewise,

$$
\sum_{k=2}^{n} \frac{1}{k} \leq \int_{1}^{n} \frac{dx}{x} = \ln n
$$

and thus

$$
\ln(n + 1) \leq \sum_{k=1}^{n} \frac{1}{x} \leq 1 + \ln n
$$