Analysis of Algorithms - Scrimmage I (Solutions)

K. Subramani
LCSEE,
West Virginia University,
Morgantown, WV
ksmani@csee.wvu.edu

Please attempt as many problems as you can in class. The scrimmage will not be graded, i.e. there are no points. The solutions are posted at:
http://www.csee.wvu.edu/~ksmani/courses/fa02/cs320/cs320.html

1. Prove using mathematical induction:

(a) \[ \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}. \]

Proof: Base case \( P(1) \):

\[
\begin{align*}
LHS &= \frac{1}{1 \cdot 2} \\
     &= \frac{1}{2} \\
RHS &= \frac{1}{1+1} \\
     &= \frac{1}{2}
\end{align*}
\]

Thus, \( LHS = RHS \) and \( P(1) \) is true.

Let us assume that \( P(k) \) is true, i.e.

\[ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{k \cdot (k+1)} = \frac{k}{k+1} \]

We need to show that \( P(k+1) \) is true.

\[
\begin{align*}
LHS &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{k \cdot (k+1)} \quad (using the inductive hypothesis) \\
     &= \frac{k}{k+1} + \frac{1}{(k+1) \cdot (k+2)} \\
     &= \frac{k}{k+1} \cdot \left( \frac{k+1}{k+2} \right) \\
     &= \frac{k+1}{k+2} \cdot \frac{(k+1)^2}{k+2} \\
     &= \frac{k+1}{k+2} \cdot \frac{(k+1)^2}{k+2}
\end{align*}
\]

\( = RHS \)
Thus, we have shown that \( P(k) \rightarrow P(k+1) \); applying the principle of mathematical induction, we conclude that the conjecture is true. \( \square \)

(b) \( 7^n - 2^n \) is divisible by 5

**Proof:** Base case \( P(1) \): Observe that \( LHS = 7^1 - 2^1 = 7 - 2 = 5 \), which is divisible by 5; therefore \( P(1) \) is true.

Let us assume that \( P(k) \) is true for some integer \( k \), i.e. \( 7^k - 2^k \) is divisible by 5 for some integer \( k \). This means \( 7^k - 2^k = 5m \) for some integer \( m \). Now consider \( P(k+1) \).

\[
LHS = 7^{k+1} - 2^{k+1} \\
= 7 \cdot 7^k - 2 \cdot 2^k \\
= 7 \cdot (5m + 2^k) - 2 \cdot 2^k \text{ (using the inductive hypothesis)} \\
= 35m + 7 \cdot 2^k - 2 \cdot 2^k \\
= 35m + 5 \cdot 2^k \\
= 5 \cdot (7m + 2^k) \\
= 5q \text{ for some } q
\]

Thus, the \( LHS \) is divisible by 5 and \( P(k+1) \) is true. We have shown that \( P(k) \rightarrow P(k+1) \); applying the principle of mathematical induction, we conclude that the conjecture is true. \( \square \)

(c) Show that \( 13^n - 6^n \) is divisible by 7.

**Proof:** Base Case: When \( n = 1 \), \( 13^n - 6^n = 13 - 6 = 7 \) is divisible by 7; thus \( P(1) \) is true.

Let us assume that \( P(k) \) is true, i.e. \( 13^k - 6^k \) is divisible by 7, for some \( k > 1 \). Thus, \( 13^k - 6^k = 7 \cdot m \), for some integer \( m \). Now consider \( P(k+1) \).

\[
13^{k+1} - 6^{k+1} = 13 \cdot 13^k - 6 \cdot 6^k \\
= 13 \cdot [6^k + 7m] - 6 \cdot 6^k \\
= 13 \cdot 7m + 13 \cdot 6^k - 6 \cdot 6^k \\
= 13 \cdot 7m + 6^k \cdot [13 - 6] \\
= 13 \cdot 7m + 6^k \cdot 7 \\
= 7 \cdot [13m + 6^k] \\
= 7 \cdot q \text{ for some integer } q
\]

Thus, applying the principle of mathematical induction, we conclude that the conjecture is true. \( \square \)

(d) Show that \( (\cos \theta + i \sin \theta)^n = \cos n \theta + i \sin n \theta \ \forall n \geq 1 \)

**Proof:** Base case \( P(1) \):

\[
LHS = (\cos \theta + i \sin \theta)^1 \\
= \cos \theta + i \sin \theta \\
= RHS
\]
Thus, $P(1)$ is true. Let us assume that $P(k)$ is true for some $k > 1$, i.e.

$$(\cos \theta + i \sin \theta)^k = \cos k\theta + i \sin k\theta$$  \hspace{1cm} (1)

Let us show that $P(k+1)$ is true.

$$(\cos \theta + i \sin \theta)^{k+1} = (\cos \theta + i \sin \theta) \cdot (\cos \theta + i \sin \theta)^k$$
\hspace{1cm} using the inductive hypothesis
\hspace{1cm} $= \cos \theta \cdot \cos k\theta + i \cos \theta \cdot \sin k\theta + i \sin \theta \cdot \cos k\theta + i^2 \sin \theta \sin k\theta$
\hspace{1cm} $= \cos \theta \cdot \cos k\theta - \sin \theta \sin k\theta + i \cdot (\cos \theta \cdot \sin k\theta + \sin \theta \cdot \cos k\theta)$
\hspace{1cm} $= \cos(1 + k)\theta + i \cdot \sin(1 + k)\theta$
\hspace{1cm} $= \text{RHS}$

We apply the principle of mathematical induction and conclude that the conjecture is true. \hfill \Box

2. Compare $f(n)$ and $g(n)$ using asymptotic notation; you may either describe $f(n)$ in terms of $g(n)$ (for instance, $f(n) = O(g(n))$) or $g(n)$ in terms of $f(n)$ (for instance, $g(n) = \omega(f(n))$). Make sure that your description is as precise as possible.

(a) $f(n) = n \log^5 n$, $g(n) = n^2$

Observe that

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{n \log^5 n}{n^2}$$
\hspace{1cm} $= \lim_{n \to \infty} \frac{\log^5 n}{n}$
\hspace{1cm} $= \lim_{n \to \infty} \frac{(5 \log^4 n) \frac{1}{n}}{1}$ (by applying L'Hopital's rule!)
\hspace{1cm} $= \lim_{n \to \infty} \frac{5 \log^4 n}{n}$
\hspace{1cm} $= \lim_{n \to \infty} \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{n}$ (by repeated application of L'Hopital's rule!)
\hspace{1cm} $= 0$

It follows that $f(n) = o(g(n))$.

(b) $f(n) = n \log_4 n$, $g(n) = n \log_5 n$.

Note that $f(n)$ can be written as $n \log_2 n \cdot \left(\frac{1}{\log_2 4}\right)$, by using the Logarithm rules. Likewise, $g(n)$ can be written as: $n \log_2 n \cdot \left(\frac{1}{\log_2 5}\right)$. It is clear that $f(n)$ and $g(n)$ differ only by a constant in their rates of growth and hence $f(n) = \Theta(g(n))$.

(c) $f(n) = \log^3 n$, $g(n) = n^\frac{1}{4}$.

By repeated application of L'Hopital's rule, we see that

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

Thus, $f(n) = o(g(n))$.

(d) $f(n) = 2^n$, $g(n) = 2^{n+1}$.

Observe that $f(n) = \frac{1}{2}g(n)$ and $g(n) = 2 \cdot f(n)$, i.e. the two functions differ at most by a constant in their rates of growth. It follows that $f(n) = \Theta(g(n))$. 

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