Proof of Fibonacci Sequence closed form

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1 Fibonacci Sequence

The Fibonacci sequence is defined as follows:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_i &= F_{i-1} + F_{i-2},\ i \geq 2 \end{aligned} \quad (1)$$

The goal is to show that

$$F_n = \frac{1}{\sqrt{5}}[p^n - q^n] \quad (2)$$

where

$$\begin{aligned} p &= \frac{1+\sqrt{5}}{2}, \text{ and} \\ q &= \frac{1-\sqrt{5}}{2}. \end{aligned} \quad (3)$$

Observe that substituting $n = 0$, gives 0 as per Definition 1 and 0 as per Formula 2; likewise, substituting $n = 1$, gives 1 from both and hence, the base cases hold.

Before we proceed, with the inductive step, we need the following identities, which you should prove.

$$\begin{aligned} 1+p &= p^2 \\ 1+q &= q^2 \end{aligned} \quad (4)$$

Now, observe that

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} \text{ (by Definition 1)} \\ &= \frac{1}{\sqrt{5}}[p^{n-1} - q^{n-1}] + \frac{1}{\sqrt{5}}[p^{n-2} - q^{n-2}] \text{ (by inductive hypothesis)} \\ &= \frac{1}{\sqrt{5}}[p^{n-1} + p^{n-2}] - \frac{1}{\sqrt{5}}[q^{n-1} + q^{n-2}] \\ &= \frac{1}{\sqrt{5}}p^{n-2}[1+p] - \frac{1}{\sqrt{5}}q^{n-2}[1+q] \\ &= \frac{1}{\sqrt{5}}p^{n-2}[p^2] - \frac{1}{\sqrt{5}}q^{n-2}[q^2] \text{ (using Identity 4)} \\ &= \frac{1}{\sqrt{5}}p^n - \frac{1}{\sqrt{5}}q^n \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{5}}[p^n - q^n] \\
&= RHS
\end{aligned}
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