1 Problems

1. Let $A$ and $B$ be propositions. Argue that the following two statements are tautologies:

   (a) $A \rightarrow A$,
   (b) $[A \land (A \rightarrow B)] \rightarrow B$

   Solution: We use truth-tables to establish the above tautologies.

   \[
   \begin{array}{c|c}
   A & A \rightarrow A \\
   \hline
   T & T \\
   F & T \\
   \end{array}
   \]

   (a)

   \[
   \begin{array}{c|c|c|c|c}
   A & B & A \rightarrow B & A \land (A \rightarrow B) & [A \land (A \rightarrow B)] \rightarrow B \\
   \hline
   T & T & T & T & T \\
   T & F & F & F & T \\
   F & T & T & F & T \\
   F & F & T & F & T \\
   \end{array}
   \]

   (b)

2. Explain the difference between the converse of a theorem and its contra-positive.

   Solution: A typical theorem has the form: *If $A_1$, then $A_2$,* where $A_1$ and $A_2$ are boolean propositions. The contra-positive of a theorem is a restatement of the above theorem as: *If not $A_2$, then not $A_1,*

   The converse of the above theorem, on the other hand is: *If $A_2, then $A_1,*

   Observe that the contra-positive of a theorem is always true. However, depending upon the theorem in question, its converse may or may not be true.

3. Use mathematical induction to show that $7^n - 2^n$ is divisible by 5, for all $n \geq 0$.

   Solution: Let $S(n)$ denote the proposition that $7^n - 2^n$ is divisible by 5, for all $n \geq 0$.

   Observe that $S(0)$ is the statement that $7^0 - 2^0 = 1 - 1 = 0$ is divisible by 5. Since every number other than zero divides zero, it follows that 5 divides 0 and $S(0)$ is true.
Assume that $S(k)$ is true, for some $k > 0$, i.e., assume that $7^k - 2^k$ is divisible by 5. Accordingly, we can write,

$$7^k - 2^k = 5 \cdot m,$$

where $m$ is some integer. (Note that this is the meaning of divisibility in the first place!)

Now we have to prove that $S(k+1)$ is true.

Observe that,

$$7^{k+1} - 2^{k+1} = 7 \cdot 7^k - 2 \cdot 2^k$$

(from (1))

$$= 7 \cdot (2^k + 5 \cdot m) - 2 \cdot 2^k$$

$$= 7 \cdot 2^k + 7 \cdot (5 \cdot m) - 2 \cdot 2^k$$

$$= (7 - 2) \cdot 2^k + 5 \cdot (7 \cdot m)$$

$$= 5 \cdot 2^k + 5 \cdot (7 \cdot m)$$

$$= 5 \cdot q,$$

where $q = 2^k + 5 \cdot (7 \cdot m)$ is an integer, since $k$ and $m$ are integers. It follows that $7^{k+1} - 2^{k+1}$ is divisible by 5, i.e., $S(k+1)$ is true. □

4. Let $S$ and $T$ denote two sets, which are subsets of a set $U$. Let $S'$ and $T'$ denote the complements of $S$ and $T$ in $U$ respectively. Prove the following set equivalence:

$$(S \cup T)' = S' \cap T'.$$

**Solution:** Let $E$ denote the set $(S \cup T)'$ and let $F$ denote the set $S' \cap T'$.

Observe that,

$$x \in E \quad \rightarrow \quad x \in (S \cup T)'$$

$$\quad \rightarrow \quad x \notin (S \cup T)$$

$$\quad \rightarrow \quad x \notin S \text{ and } x \notin T$$

$$\quad \rightarrow \quad x \in S' \text{ and } x \in T'$$

$$\quad \rightarrow \quad x \in S' \cap T'$$

$$\quad \rightarrow \quad x \in F$$

Likewise, observe that,

$$x \in F \quad \rightarrow \quad x \in S' \cap T'$$

$$\quad \rightarrow \quad x \in S' \text{ and } x \in T'$$

$$\quad \rightarrow \quad x \in S' \text{ and } x \in T'$$

$$\quad \rightarrow \quad x \notin S \text{ and } x \notin T$$

$$\quad \rightarrow \quad x \notin S \cup T$$

$$\quad \rightarrow \quad x \notin (S \cup T)$$

$$\quad \rightarrow \quad x \in (S \cup T)'$$

$$\quad \rightarrow \quad x \in E$$

It therefore follows that the set equivalence holds. □
5. Let $\Sigma = \{0, 1\}$ denote an alphabet. Enumerate five elements of the following languages:

(a) Even binary numbers,
(b) The number of zeros is not equal to the number of ones in a binary string.

Solution:

(a) $L = \{0, 10, 100, 1000, 1010\}$

(b) $L = \{1, 100, 101, 110, 010\}$. □