Fractional Knapsack

K. Subramani
LCSEE,
West Virginia University,
Morgantown, WV
{ksmani@csee.wvu.edu}

1 Statement of Problem

In the Fractional Knapsack problem, you are given \( n \) objects \( O = \{o_1, o_2, \ldots, o_n\} \) with respective weights \( W = \{w_1, w_2, \ldots, w_n\} \) and respective profits \( P = \{p_1, p_2, \ldots, p_n\} \). The goal is to pack these objects into a knapsack of capacity \( M \), such that the profit of the objects in the knapsack is maximized, while the weight constraint is not violated. You may choose a fraction of an object, if you so decide; if \( \alpha_i \), \( 0 \leq \alpha_i \leq 1 \) of object \( o_i \) is chosen, then the profit contribution of this object is \( \alpha_i \cdot o_i \) and its weight contribution is \( \alpha_i \cdot w_i \). Design a greedy algorithm for this problem and argue its correctness.

2 Solution

The solution technique consists of the following steps:

(i) Order the objects by profit per unit weight, so that \( \frac{p_1}{w_1} \geq \frac{p_2}{w_2} \geq \ldots \geq \frac{p_n}{w_n} \).

(ii) Process the objects from \( o_1 \) to \( o_n \). Pack as much as possible of \( o_1 \) in the knapsack. If the knapsack is full stop; otherwise, \( o_1 \) is included as a whole and there is weight capacity left over. Then pack as much as possible of \( o_2 \) in the knapsack and so on.

Let \( X = \langle x_1, x_2, \ldots, x_n \rangle \) denote the greedy solution vector, where \( x_i, 0 \leq x_i \leq 1 \) is the fraction of \( o_i \) that is included in the knapsack. As per the description of the greedy algorithm, 0 or more of the \( x_i \)'s will be 1, followed by a fractional quantity, followed by 0s. Let \( j \) be the first index such that \( x_j \neq 1 \). Then \( x_i = 1, i = 1, 2, \ldots, j-1 \) and \( x_i = 0, i = j + 1, j + 2, \ldots, n \). Let \( Y = \langle y_1, y_2, \ldots, y_n \rangle \) denote an arbitrary optimal solution vector. We will show that \( Y \) can be gradually transformed into \( X \), without decreasing profitability, while maintaining feasibility.

We assume that \( \sum_{i=1}^n w_i \cdot y_i = M \), since otherwise, we could pack more (of) objects into the knapsack, thereby proving that \( Y \) is sub-optimal. From the mechanics of the greedy algorithm, either \( \sum_{i=1}^n w_i \cdot x_i = M \) or \( X = \langle 1, 1, \ldots, 1 \rangle \). In the latter case, \( X \) must be optimal, so there is nothing to be proved.

Let \( k \) be the first index, where \( x_k \neq y_k \). It must be the case that \( x_k > y_k \). If \( k < j \), then \( x_k = 1 \) and \( x_k \neq y_k \) implies \( y_k < x_k \). If \( k \geq j \) and \( y_k > x_k \), then \( \sum_{i=1}^n w_i \cdot y_i > M \), and knapsack feasibility is violated.

Now increase \( y_k \) till it becomes \( x_k \), while decreasing some or all of the \( y_i, i = k + 1, \ldots, n \), so that the total weight in the knapsack stays the same. Let \( Z = \langle z_1, z_2, \ldots, z_n \rangle \) denote the new solution. Observe that \( w_k \cdot (z_k - y_k) = \sum_{i=k+1}^n w_i \cdot (y_i - z_i) \), in order to maintain feasibility.

Now,

\[
\sum_{i=1}^n p_i \cdot z_i = \sum_{i=1}^n p_i \cdot y_i + p_k \cdot (z_k - y_k) - \sum_{i=k+1}^n p_i \cdot (y_i - z_i)
= \sum_{i=1}^n p_i \cdot y_i + p_k \cdot (z_k - y_k) \cdot \frac{w_k}{w_k} - \sum_{i=k+1}^n p_i \cdot (y_i - z_i) \cdot \frac{w_i}{w_i}
\]
\[
\geq \sum_{i=1}^{n} p_i \cdot y_i + \frac{p_k}{w_k} \cdot (z_k - y_k) \cdot w_k - \sum_{i=k+1}^{n} \frac{p_k}{w_k} \cdot (y_i - z_i) \cdot w_i \\
= \sum_{i=1}^{n} p_i \cdot y_i + \frac{p_k}{w_k} \cdot [(z_k - y_k) \cdot w_k - \sum_{i=k+1}^{n} w_i \cdot (z_i - y_i)] \\
= \sum_{i=1}^{n} p_i \cdot y_i
\]

Thus, \( Z \) is one step closer to \( X \) than \( Y \) is; arguing in this fashion, we can gradually transform \( Y \) into \( X \), while maintaining feasibility and not decreasing profitability. This proves that the greedy solution is optimal.