Notes from lecture on Polyhedra, February 1, 2001
David Owen

1 Definitions

Linear Subspace

The set (of vectors) S is called a "linear subspace" if it closed under both addition and scalar multiplication. For example, if the vectors $\vec{a}$ and $\vec{b}$ are in the set *S*, and δ and μ are constants, the new vector $\vec{v} = \delta\vec{a} + \mu\vec{b}$ must also be in the set *S*, and this is true for any constant values assigned to δ and μ.

$\vec{a}, \vec{b} \in S; \forall \delta, \mu : \delta\vec{a} + \mu\vec{b} \in S$ [KV00]

The simplest linear subspace is the set including only $\vec{0}$: { $\vec{0}$ }. Another simple example of a linear subspace is a line through the origin in 2-dimensional space. In general, any set of solutions (valid x values) to the system $\mathbf{A}\vec{x} = \vec{0}$ form a linear subspace, where **A** is an $m \times n$ matrix in *n*-dimensional space.

Dimension

The "dimension" of an arbitrary set of vectors *S* is equal to the number of linearly independent vectors possible in that set. For example, if the set *S* is just one point, its dimension is 0 (by definition–see the definition of dim X below); if *S* is a line, its dimension is 1 because any line has only 1 linearly independent vector; the dimension of a plane, which has 2 linearly independent vectors, is 2; etc. In general, the dimension of a nonempty set of vectors *X* in *n*-dimensional space,

dim $X = n$ - max{rank(**A**): **A** is an $n \times n$ matrix, and $\mathbf{A}\vec{x} = \mathbf{A}\vec{y} \;\; \forall \vec{x}, \vec{y} \in X$ }

For example, if *X* is a line in 2-dimensional space, and the vectors $\vec{x} = \langle 1,1 \rangle$ and $\vec{y} = \langle 2,2 \rangle$ are on that line, the maximum rank of a matrix **A** for which the equality $\mathbf{A}\vec{x} = \mathbf{A}\vec{y}$ holds is equal to 1, and the dimension of *X* (a line in 2-dimensional space) is equal to 2 - rank(**A**) = 2 - 1 = 1. [KV00]

Affine Subspace

An "affine subspace" is a linear subspace translated by a vector–i.e. it is of the form $\mathbf{A} = \{\vec{u} + S : \vec{u}$ is an arbitrary vector and *S* is a linear subspace}. It can also be thought of as the set of solutions to the System $\mathbf{A}\vec{x} = \vec{b}$, where $\vec{b}$ is not necessarily $\vec{0}$ (if $\vec{b}$ were $\vec{0}$, the set would also be a linear subspace). For example, a line in 2-dimensional space that doesn't pass through the origin would not be a linear subspace, but it would be an affine subspace. The dimension of an affine subspace is equal to the dimension of the linear subspace formed by substituting $\vec{0}$ for $\vec{b}$.

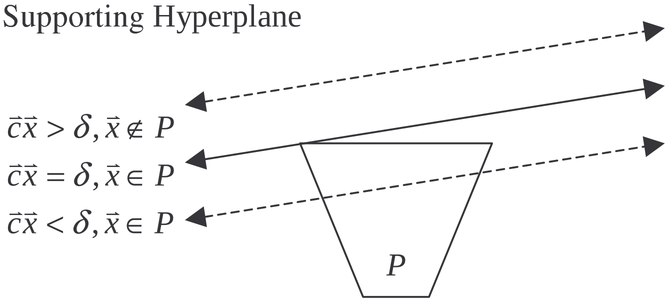
If the set X is an affine subspace (a set of solutions to $\mathbf{A}\vec{x} = \vec{b}$), and the set X' is the corresponding linear subspace (a set of solutions to $\mathbf{A}\vec{x} = \vec{0}$), the dimension

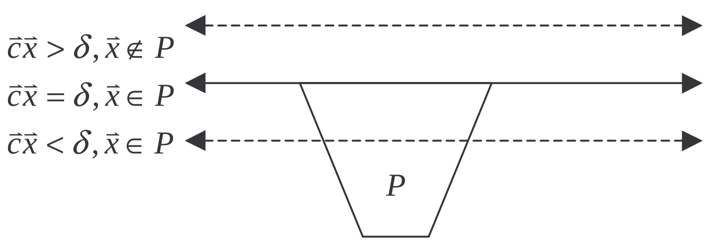
dim X = dim X' = n - max {rank($\mathbf{A}$): $\mathbf{A}$ is an $n \times n$ matrix, and $\mathbf{A}\vec{x} = \mathbf{A}\vec{y}\ \ \forall \vec{x}, \vec{y} \in X$ }

Full-Dimensional

A set X in n-dimensional space is "full-dimensional" if dim $X = n$. [KV00] Any set that is not full-dimensional will have no interior point. For example, in 2-dimensional space a line is not full-dimensional, because any point on the line is on the line's exterior. But a half-space formed by the inequality bounded by the line would be full-dimensional, because a half-space has many interior points.

Supporting Hyperplane



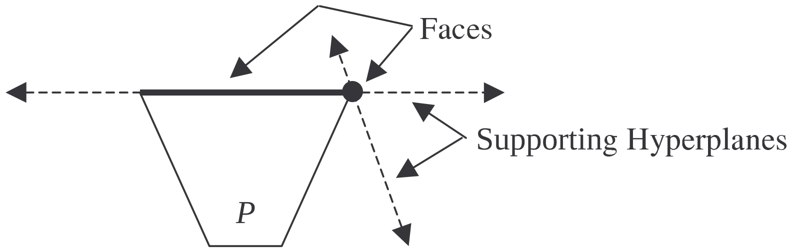


A "supporting hyperplane" of a polyhedron is a hyperplane that touches the outside of the polyhedron. If we consider a polyhedron P in 2-dimensional space, any line along an edge of P or any line outside of P that touches one of P's corners is a supporting hyperplane of P. In general, a non-zero vector $\vec{c}$ is a supporting hyperplane of P if

$\delta = \max\{\vec{c}\vec{x} : \vec{x} \in P\}$ [KV00]

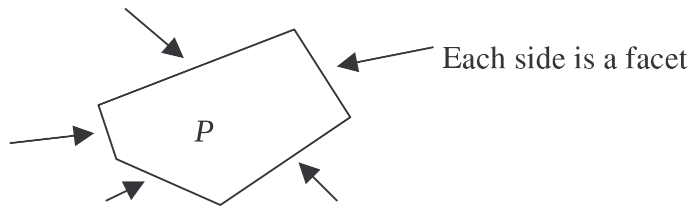
If we think in terms of the optimal solution to a linear program, $\vec{c}\vec{x} = \delta$ is the equality for which the objective function is maximized.

Face



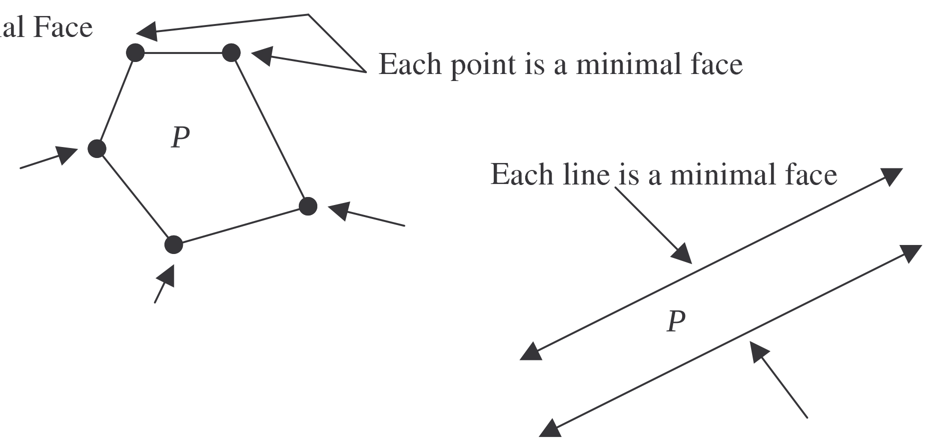
A "face" of a polyhedron *P* is either *P* itself of the intersection of *P* with one of *P*'s supporting hyperplanes. [KV00] For example, if there is a bounded polyhedron *P* in 2-dimensional space, its faces include *P* (itself), the lines defining the boundary of *P*, and the points forming the corners of *P*.

Facet (or Maximal Face)



A "facet," or "maximal face" of a polyhedron *P* is a face of *P* that is not included inside any other face, excluding the face that is *P* itself. [KV00] If *P* is again a bounded polyhedron in 2-dimensional space, its facets are the lines defining the boundary of *P*, but not the points on *P*'s corners, because these points would already be included in the lines defining the boundary of *P*.

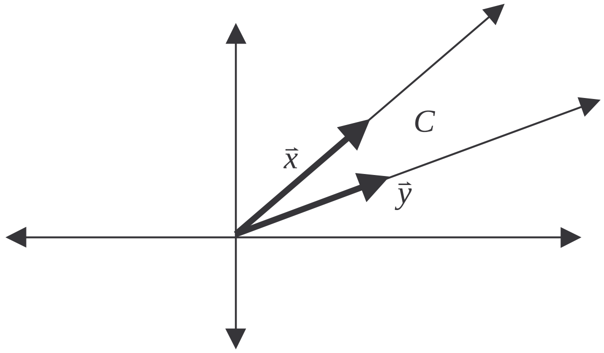
Minimal Face



A "minimal face" of a polyhedron *P* is a face that does not include any other face but itself. If *P* is again a bounded polyhedron in 2-dimensional space, its minimal faces are the points defining its corners. In most cases minimal faces will be points; however, it is possible for unbounded polyhedrons to have no corners, in which case the minimal faces may not be points.

Cone

A "cone" is a set $C \subseteq \Re^n$ for which $\vec{x}, \vec{y} \in C$ and $\lambda, \mu \geq 0$ implies $\lambda\vec{x} + \mu\vec{y} \in C$. [KV00]



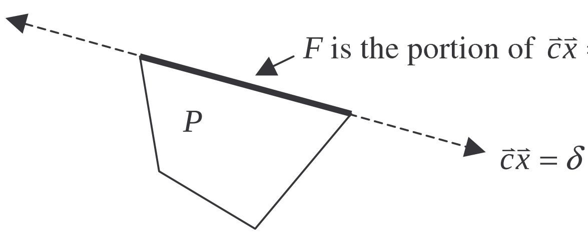
2 Theorem

The following 3 statements are equivalent.

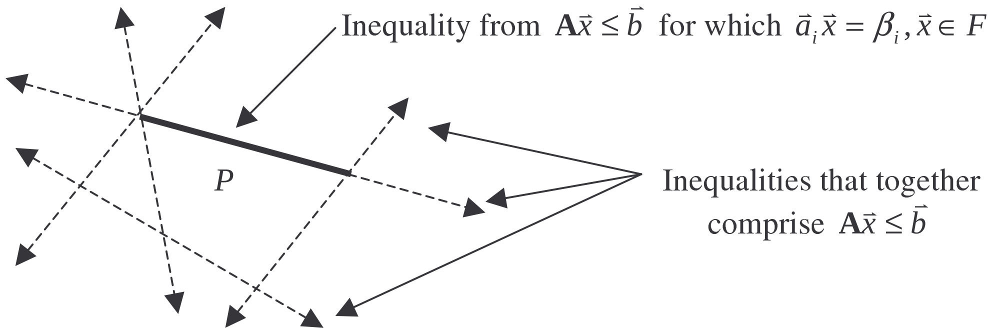
a) F is a face of P
b) There exists a vector $\vec{c}$ such that $\delta = \max\{\vec{c}\vec{x} : \vec{x} \in P\}$ and $F = \{\vec{x} \in P : \vec{c}\vec{x} = \delta\}$
c) $F = \{\vec{x} \in P : \mathbf{A}'\vec{x} = \vec{b}'\} \neq 0$ for some subsystem $\mathbf{A}'\vec{x} \leq \vec{b}'$ of $\mathbf{A}\vec{x} \leq \vec{b}$

Discussion

It should be clear from the above definitions for supporting hyperplanes and for faces that a) and b) are equivalent. b) and c) represent two different ways of understanding what it means for F to be a face of P. For b), if we think in terms of a linear program, with the polyhedron P representing all feasible solutions, a face of P is a set of solutions for which some objective function $\vec{c}\vec{x}$ is maximized.



For c), if we think of the set of linear inequalities $\vec{a}_i\vec{x} \leq \beta_i$ (where i = 1...n) that together comprise the system $\mathbf{A}\vec{x} \leq \vec{b}$, a face of the polyhedron representing solutions to $\mathbf{A}\vec{x} \leq \vec{b}$ is a subset of the polyhedron for which some of those linear inequalities are strictly equal (and most likely others aren't).



For the proof we will assume that a) and b)'s equivalence is self-evident and show only c) $\Rightarrow$ b) and b) $\Rightarrow$ c).

Proof

c) $\Rightarrow$ b)

From c) we are given a face *F*, which is a subset of the polyhedron *P* described by the system of linear inequalities $\mathbf{A}\vec{x} \le \vec{b}$. We know that for a subsystem $\mathbf{A}'\vec{x} \le \vec{b}'$ of the system $\mathbf{A}\vec{x} \le \vec{b}$, equalities $\mathbf{A}'\vec{x} = \vec{b}'$ (corresponding to the inequalities in that subsystem) hold for all *x* values in the set *F* (the face). We need to show that some vector $\vec{c}$ and some constant value δ exist so that

$F = \{\vec{x} \in P : \vec{c}\vec{x} = \delta\}$ and $\delta = \max\{\vec{c}\vec{x} : \vec{x} \in P\}$ (from b) above)

For $\vec{c}$, we make a new vector by summing all the rows in $\mathbf{A}'$, and for δ we add up all the components of $\vec{b}'$. We should find that $\vec{c}\vec{x} \le \delta$ for all *x* values in *P*, and that $\vec{c}\vec{x} = \delta$ for all *x* values in *F*. [KV00] (Make sure that you understand this technique. This is true only for equalities. In general it is not true for inequalities!)

b) $\Rightarrow$ c)

From b) we are given a face *F*, which is a subset of the polyhedron *P*, and a vector $\vec{c}$, so that

$F = \{\vec{x} \in P : \vec{c}\vec{x} = \delta\}$ and $\delta = \max\{\vec{c}\vec{x} : \vec{x} \in P\}$ (as is stated above).

$\mathbf{A}\vec{x} \le \vec{b}$ is the system of inequalities that together describe *P*. We need to show that for *F* at least 1 of these inequalities is strictly equal (there may be more than one inequality from the system that is strictly equal; there will likely be several inequalities that are not strictly equal).

Again, we are trying to show that for some subsystem from $\mathbf{A}\vec{x} \le \vec{b}$, which we call $\mathbf{A}'\vec{x} \le \vec{b}'$, the inequalities in the subsystem are strictly equal for the values of *x* in the face *F*; that is, $\mathbf{A}'\vec{x} = \vec{b}'$ for all $x \in F$. In fact we make $\mathbf{A}'\vec{x} \le \vec{b}'$ the maximal subsystem for which $\mathbf{A}'\vec{x} = \vec{b}'$, so that for any other inequalities in the general system $\mathbf{A}\vec{x} \le \vec{b}$ (we call the subsystem of these other inequalities $\mathbf{A}''\vec{x} \le \vec{b}''$), $\mathbf{A}''\vec{x} \ne \vec{b}''$ for all $x \in F$. [KV00]

At this point we have broken the system $\mathbf{A}\vec{x} \le \vec{b}$ into two subsystems, $\mathbf{A}'\vec{x} \le \vec{b}'$ and $\mathbf{A}''\vec{x} \le \vec{b}''$. For each linear inequality $\vec{a}_i'\vec{x} \le \beta_i'$ in the $\mathbf{A}'\vec{x} \le \vec{b}'$ subsystem, $\vec{a}_i'\vec{x} = \beta_i'$ for all $x \in F$; for each inequality $\vec{a}_i''\vec{x} \le \beta_i''$ in the $\mathbf{A}''\vec{x} \le \vec{b}''$ subsystem, $\vec{a}_i''\vec{x} \ne \beta_i''$ for all $x \in F$.

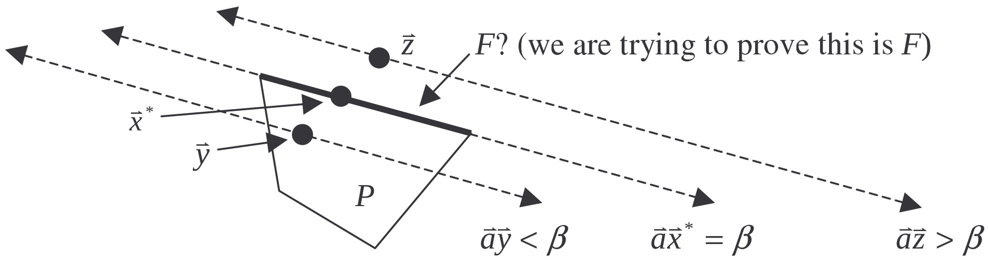
For each of the inequalities in the $\mathbf{A}''\vec{x} \le \vec{b}''$ subsystem, we know that $\vec{a}_i''\vec{x} \le \beta_i''$ for *x* values inside the polyhedron *P*. And we have chosen inequalities for this subsystem so that, as is stated above, $\vec{a}_i''\vec{x} \ne \beta_i''$ for *x* values in the face *F*. From these two facts we know that for each inequality $\vec{a}_i''\vec{x} \le \beta_i''$, there must be a point $\vec{x}_i \in F$ for which $\vec{a}_i''\vec{x} < \beta_i''$ (it's either less than or equal; it's not equal; therefore it's strictly less than).

So we have some number *k* points in the face: one for each inequality in the $\mathbf{A}''\vec{x} \le \vec{b}''$ subsystem. We know that each of these points is strictly less than at least one of the inequalities. We would like to have just one point that we know is strictly less than all of the inequalities. For this we use the center of gravity (an *n*-dimensional average) of all the points, $\vec{x}^* = \frac{1}{k}\sum_{i=1}^{k} \vec{x}_i$. We know that $\vec{x}^* \in F$ and $\vec{a}_i''\vec{x}^* < \beta_i''$ for every inequality in the $\mathbf{A}''\vec{x} \le \vec{b}''$ subsystem. [KV00]

We want to prove that $\mathbf{A}'\vec{y} = b'$ cannot hold for any $\vec{y} \in P \setminus F$ (in the polyhedron but not in the face). We know that $\vec{c}\vec{y} < \delta$ for all $\vec{y} \in P \setminus F$; that is, for any point that is in the polyhedron but not in the face, we know that the value of $\vec{c}\vec{y}$ is not maximized–in a sense $\vec{y}$ is "behind" the face, in the interior of the polyhedron. We construct a point $\vec{z}$ from $\vec{x}^*$ and $\vec{y}$: $\vec{z} = \vec{x}^* + \varepsilon(\vec{x}^* - \vec{y})$ for some small $\varepsilon > 0$. This should give us a point on the opposite side of $\vec{x}^*$ from $\vec{y}$.

ε can be any positive number. In order to make the rest of the proof work out, we choose ε to be smaller than $\dfrac{\beta_i'' - \vec{a}_i''\vec{x}^*}{\vec{a}_i''(\vec{x}^* - \vec{y})}$ for all $i \in \{1...k\}$ with $\vec{a}_i''\vec{x}^* > a_i''\vec{y}$. Since we know that $\vec{a}_i''\vec{x}^* \le \beta_i''$, we know that the numerator has to be positive. And since we only pick those denominators that are positive ($\vec{a}_i''\vec{x}^* > a_i''\vec{y}$), we know that ε is positive. [KV00]

We know that $\vec{c}\vec{y} < \delta$ (since $\vec{y}$ is not in the face). We know that $\vec{c}\vec{x}^* = \delta$ because $\vec{x}^*$ is in the face. And since ε is a positive number we know that $\vec{c}\vec{z} > \delta$. This means $\vec{z}$ is outside the polyhedron *P*, which makes sense because $\vec{z}$ is on the opposite side of $\vec{x}^*$ from $\vec{y}$, and we know $\vec{x}^*$ is in the face and $\vec{y}$ is inside the polyhedron. Since $\vec{z}$ is outside the polyhedron, we know that $\vec{z}$ must violate at least one of the inequalities $\vec{a}\vec{x} \le \beta$ from the system $\mathbf{A}\vec{x} \le \vec{b}$; so for the inequality(s) that $\vec{z}$ violates, $\vec{a}\vec{z} > \beta$.



So for at least one of the system's inequalities $\vec{a}\vec{x} \le \beta$, we know that $\vec{a}\vec{z} > \beta$. This forces $\vec{a}\vec{x}^* > \vec{a}\vec{y}$ (substitute in the defining equation for z). We have chosen ε in such a way that this inequality $\vec{a}\vec{x} \le \beta$ can not be in the subsystem $\mathbf{A}''\vec{x} \le \vec{b}''$:

$$\vec{z} = \vec{x}^* + \varepsilon(\vec{x}^* - \vec{y})$$
$$\vec{a}\vec{z} = \vec{a}\vec{x}^* + \varepsilon\vec{a}(\vec{x}^* - \vec{y})$$

Since $\varepsilon < \dfrac{\beta - \vec{a}\vec{x}^*}{\vec{a}(\vec{x}^* - \vec{y})}$ for all β and $\vec{a}$ from the $\mathbf{A}''\vec{x} \le \vec{b}''$ subsystem,

$$\vec{a}\vec{z} < \vec{a}\vec{x}^* + \frac{\beta - \vec{a}\vec{x}^*}{\vec{a}(\vec{x}^* - \vec{y})}(\vec{a}(\vec{x}^* - \vec{y}))$$
$$\vec{a}\vec{z} < \vec{a}\vec{x}^* + \beta - \vec{a}\vec{x}^*$$

$\vec{a}\vec{z} < \beta$, which contradicts our earlier conclusion that $\vec{a}\vec{z} > \beta$. [KV00]

The inequality $\vec{a}\vec{x} \le \beta$ is not in the subsystem $\mathbf{A}''\vec{x} \le \vec{b}''$; therefore it must be in the other subsystem $\mathbf{A}'\vec{x} \le \vec{b}'$. This means there is at least one inequality from the overall system $\mathbf{A}\vec{x} \le \vec{b}$ that holds to be strictly equal for the face *F*. So the subsystem $\mathbf{A}'\vec{x} \le \vec{b}'$ exists and has at least one member; therefore statement c) above is satisfied.

[KV00] B. Korte and J. Vygen. *Combinatorial Optimization*. Number 21. Springer-Verlag, New York, 2000.