CS 491 I Approximation Algorithms

Lecture Notes

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Integer Programming:

Instance: A matrix $A \in Z^{m*n}$ and vectors $\vec{b} \in Z^m$, $\vec{c} \in Z^n$.

Task: Find a vector $\vec{x} \in Z^n$ such that $A\vec{x} \le \vec{b}$ and $\vec{c}\vec{x}$ is maximum.

The set of feasible solutions can be written as $\{\vec{x} : A\vec{x} \le \vec{b}, \vec{x} \in Z^n\}$ for some matrix A and some vector $\vec{b}$. $\{\vec{x} : A\vec{x} \le \vec{b}\}$ is a polyhedron *P*. Let us define by $P_I = \{\vec{x} : A\vec{x} \le \vec{b}\}_I$ the convex hull of integral vectors in P. P_I is called the *integer hull* of *P*.

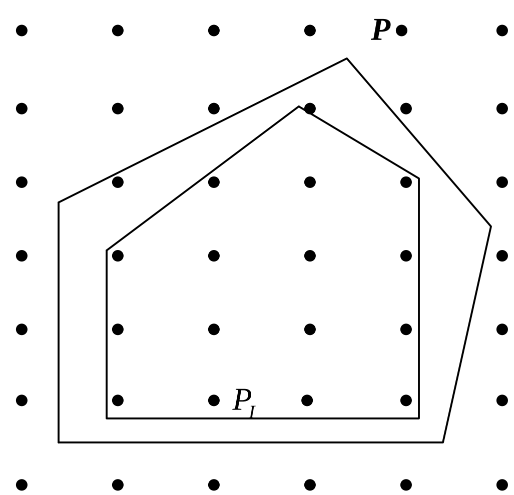

Figure 4.

Obviously $P_I \subseteq P$. If P is bounded then P_I is also a polytope

Proposition:

Let $P = \{\vec{x} : A\vec{x} \le \vec{b}\}$ be some rational polyhedron whose integer hull is nonempty, and let $\vec{c}$ be some vector. Then $\max\{\vec{c}\vec{x} : \vec{x} \in P\}$ is bounded if and only if $\max\{\vec{c}\vec{x} : \vec{x} \in P_I\}$ is bounded.

Proof: Suppose $\max\{\vec{c}\vec{x} : \vec{x} \in P\}$ is bounded. Then the dual LP $\min\{\vec{y}\vec{b} : \vec{y}A = \vec{c}, \vec{y} \ge 0\}$ is infeasible. There is a rational (and thus an integral) vector $\vec{z}$ with $\vec{c}\vec{z} < 0$ and $A\vec{z} \ge 0$. Let $\vec{y} \in P_I$ be some integral vector. Then $\vec{y} - k\vec{z} \in P_I$ for all $k \in N$ and thus $\max\{\vec{c}\vec{x} : \vec{x} \in P_I\}$ is bounded. The other direction is trivial.

Theorem: Let P be a rational polyhedron, $P = \{\vec{x} : A\vec{x} \le \vec{b}\}$. Then the following statements are equivalent:

(a) P is integral

(b) Each face of *P* contains integral vectors.

(c) Each minimal face of *P* contains integral vectors

(d) Each supporting hyperplane contains integral vectors.

(e) Each rational supporting hyperplane contains integral vectors.

(f) Max $\{\vec{c}\vec{x} : A\vec{x} \le \vec{b}\}$ is attained by an integral vector for each integral *c* for which the maximum is finite.

(g) Max $\{\vec{c}\vec{x} : A\vec{x} \le \vec{b}\}$ is an integer for each integral *c* for which the maximum is finite.

Proof:

$a \Rightarrow b$ **:** Let F be a face, $F = P \cap H$, where *H* is a supporting hyperplane, and let $\vec{x} \in F$ **.** If $P = P_I$, then $\vec{x}$ is a convex combination of integral points in *P*, and these must belong to *H* and thus to *F*.

$b \Rightarrow c$ **:** A minimal face of *P* is one of the faces of *P* which based on *b* contains integral vectors.

$c \Rightarrow d$ **:** Let F be a face and H be a supporting hyperplane, $F = P \cap H$ **.** If each minimal face contains integral vectors the supporting hyperplanes also contains integral vectors.

$d \Rightarrow e$ **:** If all supporting hyperplanes contain integral vectors, rational supporting hyperplanes contain integral vectors as well.

$e \Rightarrow f$ **:** Let $H = \{\vec{x} : \vec{c}\vec{x} = d\}$ be a rational supporting hyperplane which contains integral vectors, $\max\{\vec{c}\vec{x} : \vec{x} \in P\} = d$, is attained by an integral vector for each *c* for which the maximum is finite.

$e \Rightarrow f$

<u>Total Dual Integrality:</u>

<u>Definition:</u> A system $A\vec{x} \leq \vec{b}$ is called *Totally Dual Integral*, (**TDI**), if the minimum in the *LP* duality equation

$$\max \{\vec{c}\vec{x} : A\vec{x} \leq \vec{b}\} = \min \{y\vec{b} : \vec{y}A = \vec{c}, \vec{y} \geq \vec{0}\}$$

has an integral optimum solution $\vec{y}$ for each integral vector $\vec{c}$ for which the minimum is finite**.**

Corollary: Let $A\vec{x} \le \vec{b}$ be a *TDI*-system where *A* is rational and $\vec{b}$ is integral.

Then the polyhedron $\{\vec{x} : A\vec{x} \le \vec{b}\}$ is integral.

Totally Unimodular Matrices:

Definition: An integer matrix *A* is said to be Totally UniModular (TUM) if each sub-determinants of the matrix is {0,+1,-1}.

Theorem: Network matrices are Totally Unimodolar.

Min-Cost Flow Problem:

Let (s,t,V,E) be a flow network with underlying directed graph $G = (V,E)$, a weighting on the arcs $c_{ij} \in R^+$ for every arc $(i,j) \in E$, and a flow value $v_0 \in R^+$. The main cost flow problem is to find a feasible s-t flow of value v_0 that has minimum cost.

Theorem: The min-cost flow problem has an integral optimum if all supply-demand values are integers.

Proof: All basis have determinant {0, +1, -1}.

$$\vec{x}_B = B^{-1}\vec{b} = \frac{adj(B)}{\det(B)}\vec{b}$$

Where $adj(B)$ is the adjoint of *B*. So if *B* is unimodular and $\vec{b}$ is integer (which we always assume), $\vec{x}$ is integral.

Definition: Adjoint of a matrix A, $adj(A)$, is a matrix, whose (i, j)-element is the cofactor of the (j,i)-element of A. $adj(A).A = A.adj(A) = \det(A).I$.

When A is nonsingular: $adj(A) = \det(A).A^{-1}$.

Definition: Unimodular Matrix:

A square, integer matrix A is called unimodular (UM), if its determinant $\det(A) = \pm 1$.

The following elementary column operations to A are called *Unimodular Transformations*:

(a) Multiplying a column by –1.

(b) Exchanging two columns.

(c) Subtracting one column from another column.

Theorem: Any matrix A of rank m, can be multiplied by a unimodular matrix U, to get (B, $\vec{0}$), where B is $m * m$ non-singular.

Proof: Suppose we have found a unimodular matrix U such that $AU = (B,0)$,

$$AU = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$$

$U = (u_1, u_2, ..., u_r)$

Let $(d_1, d_2, ..., d_k)$ be the first row of D. Apply unimodular transformations such that all $d_i's$ are nonnegative and $\sum_{i=1}^{k} d_i$ is minimum. Without loss of generality, $d_1 \geq d_2 \geq ... \geq d_k$. Then $d_1 > 0$ since the rows of A (and hence those of AU) are linearly independent.

If $d_2 > 0$, then subtracting the second column of *D* from the first one would decrease $\sum_{i=1}^{k} d_i$. So $d_1 = d_2 = ... = d_k = 0$. We can increase the size of *B* by one and continue.

<u>Lemma:</u> If *U* is unimodular then U^{-1} is unimodular, further $\vec{x} \to U\vec{x}$ and $\vec{x} \to \vec{x}U$ are bijections on Z^n.

$$\det(U).\det(U^{-1}) = \det(U.U^{-1}) = \det(I) = 1$$

Integer Farkas:

Either $A\vec{x} = \vec{b}$ has an integral solution or $\vec{y}\vec{b}$ is integral for every $\vec{y}$, such that $\vec{y}A$ is integral.

<u>Lemma:</u> Let *A* be a rational matrix and $\vec{b}$ a rational column vector. Then $A\vec{x} = \vec{b}$ has an integral solution if and only if $\vec{y}\vec{b}$ is an integer for each rational vector for which $\vec{y}A$ is integral.

Necessity: If $\vec{x}$ and $\vec{y}A$ are integral vectors and $A\vec{x} = \vec{b}$ then $\vec{y}\vec{b} = \vec{y}A\vec{x}$ is an integer.

Sufficiency: Suppose $\vec{y}\vec{b}$ is an integer whenever $\vec{y}A$ is integral.

We may assume $A\vec{x} = \vec{b}$ contains no redundant equalities, i.e. $\vec{y}A = 0$ which implies $\vec{y}\vec{b} \neq 0$ for all $\vec{y} \neq 0$. Let *m* be the number of rows of *A*. If *rank (A) < m,* then $\{\vec{y} : \vec{y}A = 0\}$ contains a non-zero vector $\vec{y}'$ and $\vec{y}'' = \frac{1}{2\vec{y}'\,\vec{b}}\vec{y}'$ satisfies $\vec{y}''A = 0$ and $\vec{y}''\,\vec{b} = \frac{1}{2} \notin Z$. So the rows of *A* are linearly independent, i.e. *rank (A) = m.*

There exist a unimodular matrix *U* with *AU = (B, 0). B* is a nonsingular $m * m$ matrix.

$B^{-1}AU = (I,0)$ is an integral matrix.

$$B^{-1} = \begin{bmatrix} \vec{y}_1 \\ \vec{y}_2 \\ . \\ . \\ . \\ \vec{y}_m \end{bmatrix}$$, for each row $\vec{y}_i$ of B^{-1}, $\vec{y}AU$ is integral, thus $\vec{y}A$ is integral. $\vec{y}\vec{b}$ is integral for each row $\vec{y}_i$ of B^{-1}. It implies that $B^{-1}\vec{b}$ is an integral vector. So $U\begin{pmatrix} B^{-1} \\ 0 \end{pmatrix}$ is an integral solution of $A\vec{x} = \vec{b}$.

<u>Hoffman and Kruskal Theorem</u>: An integral matrix *A* is totally unimodular if and only if the polyhedron $\{x : A\vec{x} \le b, x \ge 0\}$ is integral for each integral vector $\vec{b}$.

<u>Lemma:</u> *A* is *TUM* if and only if [*A*, *I*] is *TUM.*

We always can add a column with one 1 and all other zeros and the result is *TUM*.

A is *TUM* $\Rightarrow \begin{bmatrix} A & \begin{matrix} 1 \\ 0 \\ . \\ . \\ 0 \end{matrix} \end{bmatrix}$ is *TUM*. Keep adding, $\begin{bmatrix} 0 \\ . \\ . \\ 1 \\ 0 \end{bmatrix}$, e_i's, to the matrix, eventually

[*A*, *I*] $\rightarrow$ *TUM.*

Proof (H-K):

Let *A* be an $m*n$ matrix and $P = \{x : A\vec{x} \le b, x \ge 0\}$. First the necessity is proved.

Suppose that *A* is totally unimodular $\Rightarrow [A, I]$ is *TUM*. Given $\vec{b}$ is integral and $\vec{x}$ be a vertex of *P*. In order to obtain $\vec{x}$ some basis, B, from *A* is picked (B is a nonsingular

$n*n$ matrix. *B*, therefore B^{-1} are integral. Since A is TUM, $|\det(B)| = 1$. By Cramer's rule $\vec{x} = B^{-1}\vec{b}$. B^{-1} and $\vec{b}$ are integral $\Rightarrow \vec{x}$ is integral.

Now sufficiency is proved. It is supposed that the vertices of *P* are integral for each integral vector $\vec{b}$. Let A' be some nonsingular $k*k$ sub-matrix of A. We have to show that $|\det(A')| = 1$. Without Loss of Generality, we can assume that A' contains the elements of the first *k* rows and *k* columns of *A*. Notice that unimodularity is preserved under exchange of rows or columns. The following figure shows the structure of matrix $[A, I]$.

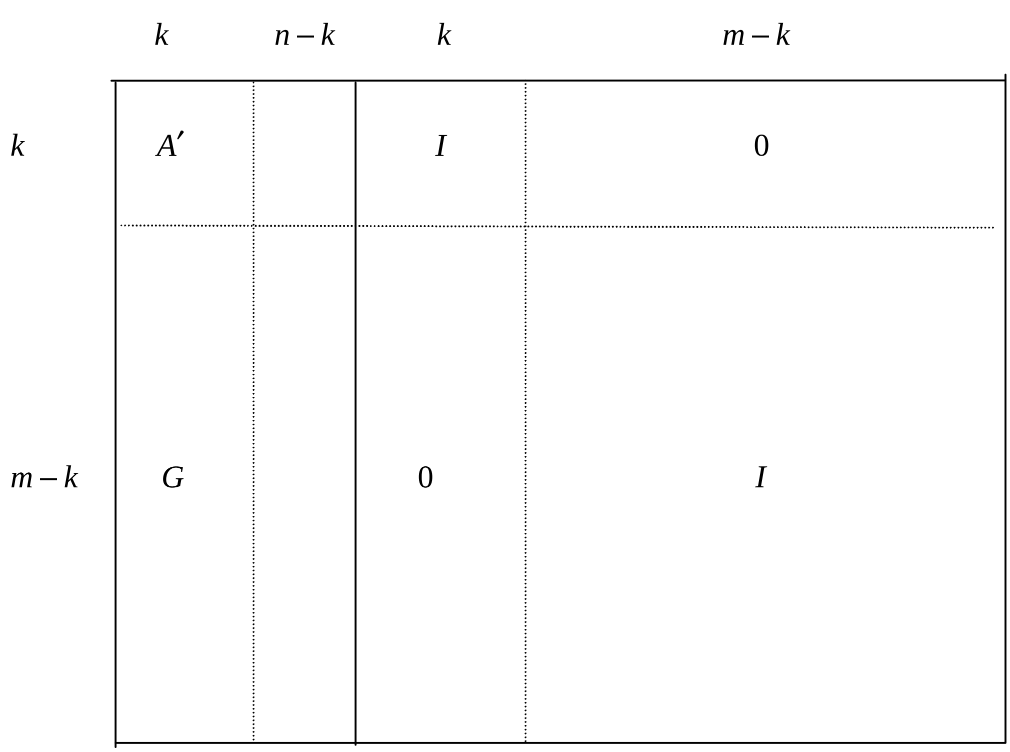

Figure 5.

Consider matrix B consisting of the first k columns and the last $m-k$ columns of A.

$$B = \begin{pmatrix} A'_{k,k} & 0_{k,m-k} \\ G_{m-k,k} & I_{m-k,m-k} \end{pmatrix}.$$

One can show that $|\det(B)| = |\det(A')|$. As we can see from Figure 6, to calculate the determinant of B, the minor matrix $M_{m,m}$ which is the matrix resulted from elimination of row and column m is multiplied by $(-1)^{k+k}$.

$|\det(B)| = 1 * M_{m,m} + 0 * M_{m-1,m} + 0 * M_{m-2,m} + \ldots + 0 * M_{m-k,m} = M_{m,m}$

One can continue by calculating the determinant of $M_{m,m}$ which is equal to $M_{m-1,m-1}$, and eventually $|\det(B)| = M_{k+1,k+1} = |\det(A')|$.

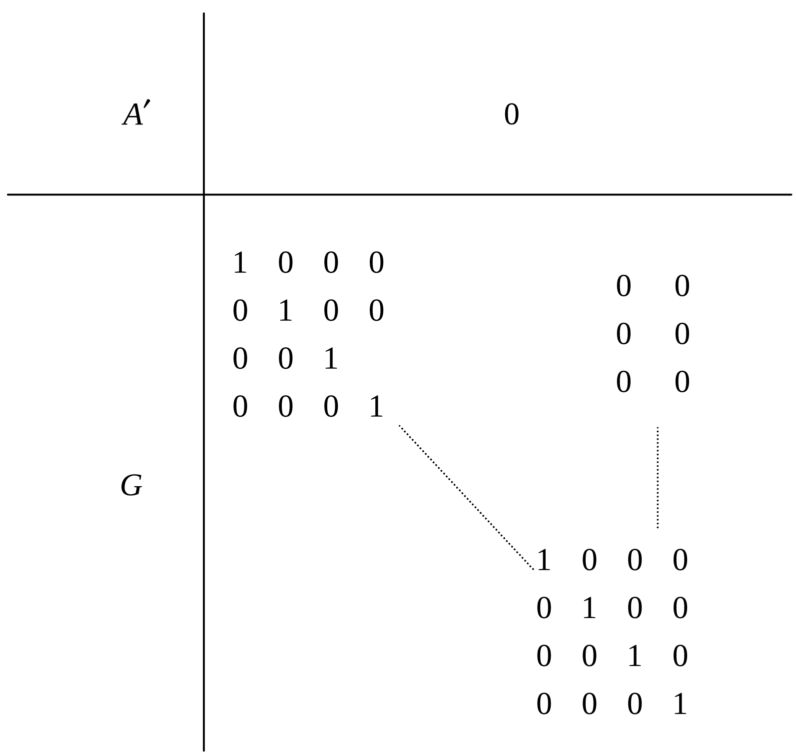

Figure 6

To prove that $|\det(B)| = 1$, we will prove that B^{-1} is integral, since $\det(B).\det(B^{-1}) = 1$, $|\det(B)| = 1$ and we are done.

Polyhedron $[A, I]\begin{bmatrix} \vec{x} \\ \vec{x}_s \end{bmatrix} \leq \vec{b}$ is integral for all integral $\vec{b}$. This means $B^{-1}\vec{b}$ is integral for all integral $\vec{b}$ and basis B. Let $i \in \{1,2,...,m\}$, let $\vec{b} = \vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ *ith* factor.

If we consider $\vec{b} = \vec{e}_1 = \begin{bmatrix} 1 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ then $B^{-1}\vec{b} = \begin{bmatrix} B_{11} & B_{21} & B_{31} & \cdots & B_{n1} \\ B_{21} & B_{22} & B_{32} & \cdots & \vdots \\ B_{31} & B_{32} & B_{33} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & B_{n3} & \cdots & B_{nn} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} B_{11} \\ B_{21} \\ B_{31} \\ \vdots \\ B_{n1} \end{bmatrix}$

The result is equal to the first column of B^{-1}, which is an integral vector so the first column of B^{-1} is integral. If we continue putting $\vec{b} = \vec{e}_i$, we can conclude that every columns of B^{-1} is an integral vector. Thus B^{-1} is comprised of *n* integral column vectors, hence B^{-1} is integral.

<u>Corollary:</u>

An integral matrix *A* is totally unimodular if and only if for all integral vectors $\vec{b}$ and $\vec{c}$ both optima in the LP duality equation are attained by integral vectors (if they are finite).

$$\max\{\vec{c}\vec{x} : A\vec{x} \leq \vec{b}, \vec{x} \geq \vec{0}\} = \min\{\vec{y}\vec{b} : \vec{y} \geq \vec{0}, \vec{y}A \geq \vec{c}\}$$

Proof: Based on H-K theorem, by using the fact that the transpose of a totally unimodular matrix is also totally unimodular.

<u>Corollary:</u>

An integral matrix A is totally unimodular if and only if the system $A\vec{x} \leq \vec{b}$, $\vec{x} \geq \vec{0}$ is **TDI** for each vector $\vec{b}$.

Proof: if *A* (and thus A^T) is TUM, then by H-K theorem $\min\{\vec{y}\vec{b} : \vec{y}A \geq \vec{c}, \vec{y} \geq \vec{0}\}$ is attained by an integral vector for each vector $\vec{b}$ and each integral vector $\vec{c}$ for which the minimum is finite. In other words, the system $A\vec{x} \leq \vec{b}, \vec{x} \geq \vec{0}$ is TDI for each vector $\vec{b}$.

To show the converse, suppose $A\vec{x} \leq \vec{b}, \vec{x} \geq 0$ is TDI for each integral vector $\vec{b}$. Then the polyhedron $\{\vec{x} : A\vec{x} \leq \vec{b}, \vec{x} \geq 0\}$ is integral for each integral vector $\vec{b}$. This means that *A* is totally unimodular.

We always can add a column with one 1 and all other zeros and the result is *TUM*.

A is $TUM \Rightarrow \begin{bmatrix} & 1 \\ & 0 \\ A & . \\ & . \\ & 0 \end{bmatrix}$ is *TUM*. Keep adding, $\begin{bmatrix} 0 \\ . \\ . \\ 1 \\ 0 \end{bmatrix}$, e_i's, to the matrix, eventually

$[A, I] \rightarrow TUM.$