Theorems and test Problems

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1 Two theorems

Theorem: 1.1 *Consider the polyhedral set* $S = \{\vec{\mathbf{x}} : \mathbf{A}.\vec{\mathbf{x}} = \vec{\mathbf{b}}, \vec{\mathbf{x}} \geq \vec{\mathbf{0}}\}$*, where* $\mathbf{A}$ *is an* $m \times n$ *rational matrix and* $rank(\mathbf{A}) = m < n$*. A point* $\vec{\mathbf{x'}} \in S$ *is an extreme point if and only if it represents the intersection of* n *linearly independent hyperplanes.*

Proof: *Let* $\vec{\mathbf{x'}}$ *be an extreme point of* S*. We need to show that it represents the intersection of* n *linerarly independent hyperplanes of* S*. Clearly* $\vec{\mathbf{x'}}$ *must satisfy the* m *constraints of* $\mathbf{A}.\vec{\mathbf{x}} = \vec{\mathbf{b}}$*. Hence it lies at the intersection of at least* m *linearly independent hyperplanes. (Remember that the rows of* $\mathbf{A}$ *are linearly independent hyperplanes since its rank is* m*.) For the remaining* $n - m$ *linearly independent hyperplanes we look at the hyperplane set* $\vec{\mathbf{x}} \geq \vec{\mathbf{0}}$*, which is a collection of* n *hyperplanes* $x_1 = 0, x_2 = 0, \ldots, x_n = 0$*. If it is the case that there are fewer than* $n - m$ *of these hyperplanes which are binding at* $\vec{\mathbf{x'}}$ *then we can write*

$$x'_i = 0, i = 1, 2, \ldots p \tag{1}$$
$$x'i > 0, i = p + 1, \ldots n \tag{2}$$
$$\tag{3}$$

where $p < n - m$*. Thus the point* $\vec{\mathbf{x'}}$ *satisfies the system*

$$\mathbf{A}.\vec{\mathbf{x}} = \vec{\mathbf{b}} \tag{4}$$
$$x_i = 0, i = 1, 2, \ldots p \tag{5}$$
$$\tag{6}$$

We can rewrite System (4) as

$$\mathbf{Q}.\vec{\mathbf{x}} = \vec{\mathbf{h}} \tag{7}$$

Observe that System (7) is a linear sysem with $m + p$ *equations and* n *variables, where* $m + p < n$*. Clearly this means that the columns of* $\mathbf{Q}$ *are linearly dependent; hence we must have a vector* $\vec{\mathbf{y}} \neq \vec{\mathbf{0}}, \in R^n$ *such that*

$$\mathbf{Q}.\vec{\mathbf{y}} = \vec{\mathbf{0}} \tag{8}$$

Now consider the two points $\vec{\mathbf{x''}}$ *and* $\vec{\mathbf{x'''}}$ *described by:*

$$\vec{\mathbf{x''}} = \vec{\mathbf{x'}} + \lambda.\vec{\mathbf{y}} \tag{9}$$
$$\vec{\mathbf{x'''}} = \vec{\mathbf{x'}} - \lambda.\vec{\mathbf{y}} \tag{10}$$
$$\tag{11}$$

$\lambda > 0$*. Now observe that* $\mathbf{Q}.\vec{\mathbf{x''}} = \mathbf{Q}.[\vec{\mathbf{x'}} + \lambda.\vec{\mathbf{y}}] = \mathbf{Q}.\vec{\mathbf{x'}}$ *and* $\mathbf{Q}.\vec{\mathbf{x'''}} = \mathbf{Q}.[\vec{\mathbf{x'}} - \lambda.\vec{\mathbf{y}}] = \mathbf{Q}.\vec{\mathbf{x'}}$ *since* $\mathbf{Q}.\vec{\mathbf{y}} = \vec{\mathbf{0}}$*. Thus the points* $\vec{\mathbf{x''}}$ *and* $\vec{\mathbf{x'''}}$ *also satisfy the system* $\mathbf{Q}.\vec{\mathbf{x}} = \vec{\mathbf{h}}$ *and hence the combined systems represented by System (4).*

Further note that, we can choose λ in such a way that $x''i = x'i + \lambda y_i \geq 0, i = p+1, \ldots n$ and $x'''i = x'i - \lambda y_i \geq 0, i = p+1, \ldots n$. This means that both $\vec{\mathbf{x''}}$ and $\vec{\mathbf{x'''}}$ belong to the set S. However, $\vec{\mathbf{x}} = \frac{1}{2}\vec{\mathbf{x''}} + \frac{1}{2}\vec{\mathbf{x'''}}$ thereby contradicting the hypothesis that $\vec{\mathbf{x'}}$ is an extreme point of S.

We now have to show the converse i.e. if $\vec{\mathbf{x'}}$ is the intersection of n linearly independent hyperplanes, it must be an extreme point. Without loss of generality, let us assume that the n linearly independent hyperplanes intersecting at $\vec{\mathbf{x'}}$ are

$$\mathbf{A}.\vec{\mathbf{x}} = \vec{\mathbf{b}} \tag{12}$$

$$x_i = 0, i = 1, 2, \ldots, n-m \tag{13}$$

$$\tag{14}$$

If $\vec{\mathbf{x'}}$ is not an extreme point, then it can be expressed in the form

$$\vec{\mathbf{x'}} = \alpha.\vec{\mathbf{x''}} + (1-\alpha).\vec{\mathbf{x'''}}, \alpha \in (0,1) \tag{15}$$

Consider the first $n-m$ coordinates of the points, $\vec{\mathbf{x'}}, \vec{\mathbf{x''}}\vec{\mathbf{x'''}}$. We must have

$$x'_i = \alpha.x''_i + (1-\alpha).x'''_i \tag{16}$$

Since both α and $(1-\alpha)$ are greater than 0, while $x'_i = 0$, x''_i and x'''_i must be 0 for $i = 1, 2, \ldots n-m$. Since, we have $\mathbf{A}.\vec{\mathbf{x'}} = \mathbf{A}.\vec{\mathbf{x''}} = \mathbf{A}.\vec{\mathbf{x'''}}$, we must have

$$\sum_{j=n-m+1}^{n} x'_j.\mathbf{a_j} = \sum_{j=n-m+1}^{n} x'_j.\mathbf{a_j} = \sum_{j=n-m+1}^{n} x'_j.\mathbf{a_j} = \vec{b} \tag{17}$$

where $\mathbf{A} = [\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}]$ i.e. the $\mathbf{a_i}$ are column vectors of $\mathbf{A}$. But the columns $\mathbf{a_{n-m+1}}, \ldots, \mathbf{a_n}$ are linearly independent since $\vec{\mathbf{x'}}$ is the unique solution to System (12). This forces $x'_i = x''_i = x'''_i$ for $i = m-n+1, \ldots n$ and we are done. □

Theorem: 1.2 *Consider the polyhedral set $S = \{\vec{\mathbf{x}} : \mathbf{A}.\vec{\mathbf{x}} = \vec{\mathbf{b}}, \vec{\mathbf{x}} \geq \vec{\mathbf{0}}\}$, where $\mathbf{A}$ is an $m \times n$ rational matrix and $rank(\mathbf{A}) = m < n$. A point $\vec{\mathbf{x'}} \in S$ is an extreme point if and only if it is a basic feasible solution.*

Proof: *First assume that $\vec{\mathbf{x'}} \in S$ is an extreme point. From the previous theorem, we know that $\vec{\mathbf{x'}}$ must lie at the intersection of n linearly independent hyperplanes. Without loss of generality, we can assume that these hyperplanes are the m hyperplanes defining $\mathbf{A}$ and $n-m$ from the set $\vec{\mathbf{x'}} \geq \vec{\mathbf{0}}$, i.e. $x'_i = 0, i = 1, 2, \ldots, n-m$. In order to show that $\vec{\mathbf{x'}}$ is a basic feasible solution, we need to show that it is feasible and basic. Since it is an extreme point, it is clearly a point of the set S and hence feasible. All that we need to show now is that $\vec{\mathbf{x'}}$ is basic. Since $n-m$ variables (components of $\vec{\mathbf{x'}}$ are set to zero, we can regard them as our vector of non-basic variables $\vec{\mathbf{x'_N}} = \vec{\mathbf{0}}$. Then $\vec{\mathbf{x}}$ is the unique solution of the n linearly independent hyperplanes $\mathbf{A}.\vec{\mathbf{x}} = \vec{\mathbf{b}}, \vec{\mathbf{x}} \geq \vec{\mathbf{0}}$. Let $\vec{\mathbf{x'_B}}$ represent the remaining m components of $\vec{\mathbf{x'}}$. We can partition $\vec{\mathbf{A}}$ to correspond to the vectors $\vec{\mathbf{x_B}}$ and $\vec{\mathbf{x'_N}}$ i.e. $\mathbf{A} = (\mathbf{B} : \mathbf{N})$. Then the extreme point $\vec{\mathbf{x'}}$ is the unique basic solution of the system $\mathbf{B}.\vec{\mathbf{x_B}} + \mathbf{N}.\vec{\mathbf{x_N}} = \vec{\mathbf{b}}$.*

Now assume that $\vec{\mathbf{x'}}$ is a basic feasible solution. This implies that there exists a basis matrix $\mathbf{B}$ such that

$$\vec{\mathbf{x'}} = \begin{pmatrix} x'_B \\ x'_N \end{pmatrix} = \begin{pmatrix} \mathbf{B^{-1}}.\vec{\mathbf{b}} \\ \vec{\mathbf{0}} \end{pmatrix} \tag{18}$$

This implies that $\vec{\mathbf{x'}}$ is the unique solution of the system $\mathbf{B}.\vec{\mathbf{x_B}} + \mathbf{N}.\vec{\mathbf{x_N}} = \vec{\mathbf{b}}$, or equivalently $\mathbf{A}.\vec{\mathbf{x}} = \vec{\mathbf{b}}, \vec{\mathbf{x_N}} = \vec{\mathbf{0}}$. Hence $\vec{\mathbf{x'}}$ lies at the intersection of n linearly independent hyperplanes and is therefore extreme. □

Definition: 1.1 *A set S is said to be convex if given two points $x_1, x_2 \in S$, the point $x_3 = \alpha.x_1 + (1-\alpha).x_2 \in S$, for all $\alpha \in [0,1]$. x_3 is said to be a convex combination of x_1 and x_2. If $\alpha > 0$, x_3 is said to be a strict convex combination.*

2 Quiz problems

1. Show that the set $\{\mathbf{A}.\vec{\mathbf{x}}\{\leq,=,\geq\}\vec{\mathbf{b}}, \vec{\mathbf{x}} \geq \vec{\mathbf{0}}\}$ is convex;

2. Solve graphically:

$$\min z = 4.x_1 + 5.x_2 \tag{19}$$
$$3.x_1 + 2.x_2 \leq 24 \tag{20}$$
$$x_1 \geq 5 \tag{21}$$
$$3.x_1 - x_2 \leq 6 \tag{22}$$
$$x_1, x_2 \geq 0 \tag{23}$$
$$\tag{24}$$

3. Show that the halfspace $\mathbf{H}^- = \{\vec{\mathbf{x}} : \vec{\mathbf{a}}.\vec{\mathbf{x}} \leq \alpha\}$ is convex.

4. Given three vectors, $\vec{\mathbf{a}} = [\mathbf{4}, \mathbf{2}]^{\mathbf{T}}, \vec{\mathbf{b}} = [-\mathbf{2}, \mathbf{6}]^{\mathbf{T}}, \vec{\mathbf{c}} = [\mathbf{2}, \mathbf{5}]^{\mathbf{T}}$, illustrate graphically

   (a) The set of all linear combinations of $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$,

   (b) The set of all non-negative linear combinations of $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$,

   (c) The set of all convex combinations of $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$.