Fourier-Motzkin Elimination

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1 The Method

Exercise 3 on Page 62 of [KV00] describes a method of solving linear programs called Fourier-Motzkin elimination. This method was discovered first by Fourier [Fou24] and then elaborated on in [DE73]. The Fourier-Motzkin elimination method is the linear programming equivalent to Gaussian elimination for solving systems of linear equations. Observe that given a system $\mathbf{A}.\vec{\mathbf{x}} = \vec{\mathbf{b}}$, where $\mathbf{A}$ is an $n \times n$ matrix of full row rank, Gaussian elimination proceeds by pivoting on each (i, i) element $i = 1, \ldots, n$. To begin with we make the element $\mathbf{A}[1, 1] = 1$ through appropriate multiplication. Then we make the co-efficent of x_1 in all rows $2, \ldots n$ zero, through appropriate scalar multiplication and addition. This process continues till we are left with an upper triangular matrix, from which the computation of the variable values is relatively straightforward. If the given system is infeasible, then we arrive at the equation $0.x_n = 1$. Also see [Str88].

Fourier observed that a similar elimination procedure could be used to solve systems of linear inequalities. In class, we argued that solving systems of linear inequalities is equivalent to linear programming in that given an oracle to decide the feasibility problem, we can construct a solution that maximizes an arbitrary linear function over the same polyhedron, taking time at most $\log z^* \times T(\mathcal{L})$, where z^* is the optimum value of the function being optimized and $T(\mathcal{L})$ is the time taken to answer a single feasibilty question. Thus Fourier's method could be used to solve linear programs. The key component of Fourier's algorithm is the following theorem.

Theorem: 1.1 *Let us say that we have a system in the form* $\mathbf{A}.\vec{\mathbf{x}} \leq \vec{\mathbf{b}}$*, where* $\mathbf{A}$ *has* m *rows and* n *columns. Without loss of generality, the system can be written in the following form:*

$$\begin{aligned} D(\vec{\mathbf{x}}): &\quad \vec{\mathbf{a_i'}}.\vec{\mathbf{x'}} \leq b_i, i = 1, \ldots, m_1 \\ E(\vec{\mathbf{x}}): &\quad -x_1 + \vec{\mathbf{a_j'}}.\vec{\mathbf{x'}} \leq b_j, j = m_1 + 1, \ldots, m_2 \\ F(\vec{\mathbf{x}}): &\quad x_1 + \vec{\mathbf{a_k'}}.\vec{\mathbf{x'}} \leq b_k, k = m_2 + 1, \ldots, m \end{aligned} \tag{1}$$

where $\vec{\mathbf{x'}}$ *is* $[x_2, x_3, \ldots, x_n]^T$ *i.e. the same set of variables without* x_1*.*

What we have done is express each constraint in the form: $x_1 \leq ()$ *(* $F(\vec{\mathbf{x}})$ *),* $x_1 \geq ()$ *(* $E(\vec{\mathbf{x}})$ *) and the constraints which do not have* x_1 *in them (* $D(\vec{x})$ *)*

Now consider the system defined below defined by:

$$\begin{aligned} D(\vec{\mathbf{x}}): &\quad \vec{\mathbf{a_i'}}.\vec{\mathbf{x'}} \leq b_i, i = 1, \ldots, m_1 \\ &\quad \vec{\mathbf{a_j'}}.\vec{\mathbf{x'}} - b_j \leq b_k - \vec{\mathbf{a_k'}}.\vec{\mathbf{x'}}, j = m_1 + 1, \ldots, m_2, k = m_2 + 1, \ldots m \end{aligned} \tag{2}$$

Then System (1) has a solution if and only if System (2) has a solution.

Proof: *Let us say that System (1) has a solution i.e. we have a vector* $\vec{\mathbf{x}} = [x_1, x_2, \ldots, x_n]$ *satisfying System (1). The value of* x_1 *chosen has to satisfy*

$$x_1 \geq b_j - \vec{\mathbf{a'_j}}.\vec{\mathbf{x'}}, \forall j = m_1 + 1, \ldots, m_2 \quad (3)$$
$$x_1 \leq \vec{\mathbf{a'_k}}.\vec{\mathbf{x'}} - b_k, \forall k = m_2 + 1, \ldots, m \quad (4)$$
$$(5)$$

Hence System (2) is trivially satisifed.

We now show the converse, i.e. if System (2) is satisifed, then System (1) is also satisfied. Consider a solution $\vec{\mathbf{x'}} = [x_2, x_3, \ldots, x_n]$ *to System (2). Let* $l = \max(\vec{\mathbf{a'_j}}.\vec{\mathbf{x'}} - b_j, j = m_1 + 1, \ldots m_2)$ *and* $u = \min(b_k - \vec{\mathbf{a'_k}}.\vec{\mathbf{x'}}, k = m_2 + 1, \ldots m)$. *If* $l > u$, *then one of the constraints of System (2) has been violated. So* $l \leq u$. *An assignment of* x_1 *to any value in the range* $[l, u]$ *trivially satisfies System (1).* □

This elimination procedure clearly gives an algorithm for deciding feasibility of a linear system of inequalities. First eliminate x_1, then x_2 and so on till you have only x_n left. If x_n occurs as a feasible range $[a, b], a < b$, then the system is feasible. Otherwise, we get $\vec{\mathbf{0}}.\vec{\mathbf{x}} \leq -1$, which is a contradiction.

One can also look at the elimination procedure as a way of projecting the input polyhedron onto successive smaller dimensional spaces, while preserving the solution space [Sch87].

2 Two examples

Example (1): Consider the problem

$$\begin{aligned} \max z = 2.x_1 + 3.x_2 \\ s.t. \\ x_1 - 2.x_2 \leq 4 \\ 2.x_1 + x_2 \leq 18 \\ x_2 \leq 10 \\ x_1, x_2 \geq 0 \end{aligned}$$

We replace the objective function with the relationship $z \leq 2.x_1 + 3.x_2$. *The idea is that to maximize* z *we are driving it to the largest possible value that can be assumed by* $2.x_1 + 3x_2$ [1] *We then have the system*

$$\begin{aligned} z - 2.x_1 - 3.x_2 \leq 0 \\ x_1 - 2.x_2 \leq 4 \\ 2.x_1 + x_2 \leq 18 \\ x_2 \leq 10 \\ x_1, x_2 \geq 0 \end{aligned}$$

To eliminate x_1, *we rewrite the system in the form of System (1).*

$$\begin{aligned} -x_1 - \frac{3}{2}.x_2 + \frac{1}{2}.z \leq 0 \\ x_1 - 2.x_2 \leq 4 \\ x_1 + \frac{1}{2}.x_2 \leq 9 \\ x_2 \leq 10 \end{aligned}$$

[1] This was the cause of confusion in class!

$$-x_1 \leq 0$$
$$-x_2 \leq 0$$

Pairing off the constraints in which x_1 appears with opposite signs, we get,

$$-\frac{3}{2}.x_2 \leq 4 + 2.x_2 \Rightarrow x_2 \geq \frac{z-8}{7}$$
$$-\frac{3}{2}.x_2 \leq 9 - 12.x_2 \Rightarrow x_2 \geq -9 + \frac{z}{2}$$
$$0 \leq 4 + 2.x_2 \Rightarrow x_2 \geq -2$$
$$0 \leq 9 - \frac{1}{2}.x_2 \Rightarrow x_2 \leq 18$$
$$x_2 \leq 10$$
$$-x_2 \leq 0$$

Now, observe that the constraint $x_2 \geq -2$ is redundant, since $x_2 \geq 0$ is already present. Likewise, $x_2 \leq 18$ is redundant, on account of the harsher constraint $x_2 \leq 10$. Accordingly, the new set of constraints is:

$$x_2 \geq \frac{z-8}{7}$$
$$x_2 \geq -9 + \frac{z}{2}$$
$$x_2 \leq 10 x_2 \geq 0$$

Pairing off constraints to eliminate x_2 we get

$$\frac{z-8}{7} \leq 10 \Rightarrow z \leq 78$$
$$-9 + \frac{z}{2} \leq 10 \Rightarrow z \leq 38$$
$$0 \leq 10$$

Since $z \leq 38$ is the more binding constraint, it the optimum value. This can be verified through graphical procedures.

Example (2): Solve the system

$$\max z = 5.x_1 + x_2$$
$$s.t.$$
$$2.x_1 + x_2 \geq 5$$
$$x_2 \geq 1$$
$$2.x_1 + 3.x_2 \leq 6$$
$$x_1, x_2 \geq 0$$

References

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