Midterm - Practice Questions

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1. Suppose my linear program \( A\mathbf{x} \leq \mathbf{b} \) has the specification that all variables except \( x_1 \) are \( \geq 0 \); \( x_1 \) is an unrestricted variable. How would you model it in the standard form with all variables \( \geq 0 \)?

2. Suppose that all \( n \) variables are unrestricted variables. Show that you can replace this set with a set of \( n + 1 \) variables that are constrained to be non-negative?

3. **Definition: 0.1** Polytope - A bounded polyhedron

   **Definition: 0.2** \( \epsilon \)-neighbourhood - The \( \epsilon \)-neighbourhood of a point \( \mathbf{x} \), denoted by \( N_\epsilon(\mathbf{x}) \), is defined as a ball of radius \( \epsilon \) centred around \( \mathbf{x} \).

   **Definition: 0.3** A point \( \mathbf{x} \) is said to be in the closure of a set \( S \), if for all \( \epsilon > 0 \), \( N_\epsilon(\mathbf{x}) \cap S \neq \emptyset \). The set of all points in the closure of \( S \) are denoted by \( \text{cl } S \). \( S \) is said to be closed if \( S = \text{cl } S \).

   Show that a polytope defined in the usual way is closed.

4. Show that the set of optimal points of a Linear Program is a convex set.

5. Can a pivot of the Simplex Algorithm move the feasible point in the basis, while leaving the cost unchanged?

6. Prove or disprove: If an LP is unbounded then there exists a vector \( \mathbf{a} \) such that for any feasible \( \mathbf{x} \), \( \mathbf{x} + k\mathbf{a} \) is also feasible, for all \( k > 0 \).

7. Use the Simplex Method to solve

   \[
   \begin{align*}
   \text{max } z &= 3x_1 + 10x_2 + 5x_3 + 11x_4 + 7x_5 + 14x_6 \\
   \text{s.t.} & \quad x_1 + 7x_2 + 3x_3 + 4x_4 + 2x_5 + 5x_6 = 42 \\
   & \quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
   \end{align*}
   \]

8. Show that the point \( \mathbf{x} = [10, 0, 16, 6] \) is an optimal solution to the problem:

   \[
   \begin{align*}
   \text{max } z &= x_1 + 2x_2 + 5x_3 + x_4 \\
   \text{s.t.} & \quad x_1 + 2x_2 + x_3 - x_4 \leq 20 \\
   & \quad -x_1 + x_2 + x_3 + x_4 \leq 12 \\
   & \quad 2x_1 + x_2 + x_3 - x_4 \leq 30 \\
   & \quad x_i \geq 0, i = 1, 2, 3, 4
   \end{align*}
   \]
9. Consider the following linear program:

$$\begin{align*}
\min z &= b^T \cdot w - c^T x \\
\text{s.t.} & & \\
A \cdot x &\leq b \\
A^T \cdot w &\leq c^T x \\
\bar{x} &\geq 0 \\
\bar{w} &\geq 0
\end{align*}$$

where $A$ is $m \times n$, $b$ is $m \times 1$, $c$ is $n \times 1$. Show that the optimal objective value is 0 or the problem is infeasible.