1 Constraint Matrices of network flow graphs

The following conventions are used to write down the constraint matrices of network flow graphs:

1. All supply is positive,
2. All demand is negative,
3. For a given node, flow entering the node is taken to be negative, while flow leaving the node is taken to be positive.

Accordingly for Figure (1), we get the following constraint set:

Figure 1: Flow Graph

\[ \text{Node 1: } y_{12} + y_{13} = 10 \]
\[ \text{Node 2: } y_{23} + y_{24} - y_{12} = 0 \]
\[ \text{Node 3: } y_{34} - y_{13} - y_{23} = -3 \]
\[ \text{Node 4: } -y_{34} - y_{24} = -7 \]

Note the quantity within the parentheses represents the supply/demand at that node.
Hence the constraint system is:

\[
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 \\
-1 & 0 & 1 & 1 & 0 \\
0 & -1 & -1 & 0 & 1 \\
0 & 0 & 0 & -1 & -1
\end{bmatrix}
\begin{bmatrix}
y_{12} \\
y_{13} \\
y_{23} \\
y_{24} \\
y_{34}
\end{bmatrix}
= \begin{bmatrix}
10 \\
0 \\
-3 \\
-7
\end{bmatrix}
\]  \( \text{(1)} \)

\[y_{ij} \geq 0 \text{ for all edges} \]  \( \text{(2)} \)

**Observation 1.1** This notation is only a convention. If you want to choose all the supply nodes to be negative and all the demand nodes to be positive, all the equations will change accordingly and flow entering a node will be positive and flow leaving a node will be negative.

**Observation 1.2** All entries in the matrix are in the set \( \{0, 1, -1\} \). Further, if you add all the rows, you get the \( \vec{b} \) vector, which indicates that the rows are linearly dependent. To apply Simplex, we add a dummy arc called root arc, from node 4 without any end-point. Then the constraint matrix becomes

\[
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 1 & 1 & 0 & 0 \\
0 & -1 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & -1 & -1
\end{bmatrix}
\begin{bmatrix}
y_{12} \\
y_{13} \\
y_{23} \\
y_{24} \\
y_{34}
\end{bmatrix}
= \begin{bmatrix}
10 \\
0 \\
-3 \\
-7
\end{bmatrix}
\]  \( \text{(3)} \)

\[y_{ij} \geq 0 \text{ for all edges} \]  \( \text{(4)} \)

## 2 Total Dual Integrality

**Definition 2.1** A system \( A \vec{x} \leq \vec{b} \) is said to be totally dual integral (TDI) if

\[
\{ \max c, \vec{x} : A \vec{x} \leq \vec{b} \} = \{ \min \bar{y}, \bar{b} : \bar{y} A = \bar{c}, \bar{y} \geq \bar{0} \}
\]  \( \text{(5)} \)

is reached at an integral \( \bar{y} \) for all integral \( \bar{c} \), where the optima are finite.

An immediate corollary of the definition and \((g) \Rightarrow (f)\) of Theorem (5.12) in [KV00] is:

**Corollary 2.1** Let \( A \vec{x} \leq \vec{b} \) be a TDI system, with \( A \) rational and \( \vec{b} \) integral. Then the polyhedron \( A \vec{x} \leq \vec{b} \) is integral.

**Proof:** Exercise. \( \Box \)

We note that being TDI or not is not a characteristic of the polyhedron under consideration, but of the linear system used to describe that polyhedron. For instance, the polyhedron in Figure (2) can be described by System (6) and System (7). However, only System (6) is TDI.

\[
\begin{bmatrix}
1 & 1 \\
1 & 0 \\
1 & -1
\end{bmatrix}
\leq
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\leq
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]  \( \text{(6)} \)

\[
\begin{bmatrix}
1 & 1 \\
1 & 0 \\
1 & -1
\end{bmatrix}
\leq
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\leq
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]  \( \text{(7)} \)

**Proof:** We first show that System (7) is not TDI. Let \( \vec{c} = [c_1, c_2] \) be an integral vector. Taking \( \{ \max c_1 x_1 + c_2 x_2 : \text{System (7)} \} \) as the primal, the dual is:

\[
\begin{align*}
\min \bar{y}.\bar{b} \\
y_1 + y_2 &= c_1 \\
y_1 - y_2 &= c_2 \\
y_1, y_2 &\geq 0
\end{align*}
\]  \( \text{(8)} \)
The only solution to this system is:
\[ y_1 = \frac{c_1}{2} \text{ and } y_2 = \frac{c_2}{2}. \] If \( c_1 \) is odd and \( c_2 \) is even, then both \( y_1 \) and \( y_2 \) are fractional, thus proving that System (7) is not TDI. (Exercise: If \( c_1 < c_2 \) the dual is infeasible. Give an explanation based on Figure (2)!) 

If the polyhedron is described by System (6), the dual is:

\[
\begin{align*}
\min y \cdot \mathbf{b} \\
y_1 + y_2 + y_3 &= c_{1} \\
y_1 - y_3 &= c_{2} \\
y_1, y_2, y_3 &\geq 0
\end{align*}
\]

The chief difference between System (9) and System (8) is that System (9) has an extra variable ( \( y_2 \) ) that allows us some freedom. We consider the following cases:

1. Both \( c_1 \) and \( c_2 \) even: Choose \( y_2 = 0 \) and you have one single integral solution for \( y_1 \) and \( y_3 \);
2. Both \( c_1 \) and \( c_2 \) odd: Choose \( y_2 = 0 \) and you have one single integral solution for \( y_1 \) and \( y_3 \);
3. \( c_1 \) odd and \( c_2 \) even: Choose \( y_2 = 1 \) and you have one single integral solution for \( y_1 \) and \( y_3 \), because \( c_1 - y_2 \) is even;
4. \( c_1 \) even and \( c_2 \) odd: Choose \( y_2 = 1 \) and you have one single integral solution for \( y_1 \) and \( y_3 \).

In all cases, we can find an integral optimum for the dual and hence the primal system is totally dual integral. □

References