Unimodularity and Total Dual Integrality

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1 Constraint Matrices of network flow graphs

The following conventions are used to write down the constraint matrices of network flow graphs:

1. All supply is positive,
2. All demand is negative,
3. For a given node, flow entering the node is taken to be negative, while flow leaving the node is taken to be postive.

Accordingly for Figure (1), we get the following constraint set:

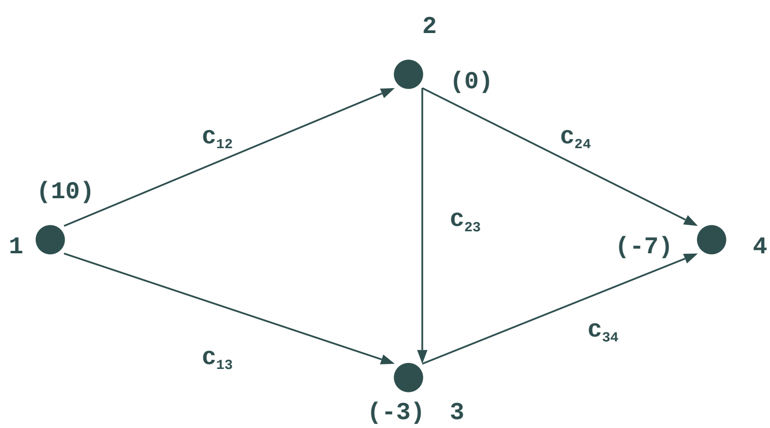

Figure 1: Flow Graph

$$Node\ 1 : y_{12} + y_{13} = 10$$
$$Node\ 2 : y_{23} + y_{24} - y_{12} = 0$$
$$Node\ 3 : y_{34} - y_{13} - y_{23} = -3$$
$$Node\ 4 : -y_{34} - y_{24} = -7$$

Note the quantity within the parentheses represents the supply/demand at that node.

Hence the constraint system is:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_{12} \\ y_{13} \\ y_{23} \\ y_{24} \\ y_{34} \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ -3 \\ -7 \end{bmatrix} \tag{1}$$

$$y_{ij} \geq 0 for\ all\ edges \tag{2}$$

Observation: 1.1 *This notation is only a convention. If you want to choose all the supply nodes to be negative and all the demand nodes to be positve, all the equations will change accordingly and flow entering a node will be positive and flow leaving a node will be negative.*

Observation: 1.2 *All entries in the matrix are in the set* $\{0, 1, -1\}$. *Further, if you add all the rows, you get the* $\vec{\mathbf{0}}$ *vector, which indicates that the rows are linearly dependent. To apply Simplex, we add a dummy arc called root arc, from node 4 without any end-point. Then the constraint matrix becomes*

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_{12} \\ y_{13} \\ y_{23} \\ y_{24} \\ y_{34} \\ y_d \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ -3 \\ -7 \end{bmatrix} \tag{3}$$

$$y_{ij} \geq 0 for\ all\ edges \tag{4}$$

2 Total Dual Integrality

Definition: 2.1 *A system* $\mathbf{A}.\vec{\mathbf{x}} \leq \vec{\mathbf{b}}$ *is said to be totally dual integral (TDI) if*

$$\{\max \vec{\mathbf{c}}.\vec{\mathbf{x}} : \mathbf{A}.\vec{\mathbf{x}} \leq \vec{\mathbf{b}}\} = \{\min \vec{\mathbf{y}}.\vec{\mathbf{b}} : \vec{\mathbf{y}}.\mathbf{A} = \vec{\mathbf{c}}, \vec{\mathbf{y}} \geq \vec{\mathbf{0}}\} \tag{5}$$

is reached at an integral $\vec{\mathbf{y}}$ *for all integral* $\vec{\mathbf{c}}$, *where the optima are finite.*

An immediate corollary of the definition and $(g) \Rightarrow (f)$ of Theorem (5.12) in [KV00] is:

Corollary: 2.1 *Let* $\mathbf{A}.\vec{\mathbf{x}} \leq \vec{\mathbf{b}}$ *be a TDI system, with* $\mathbf{A}$ *rational and* $\vec{\mathbf{b}}$ *integral. Then the polyhedron* $\mathbf{A}.\vec{\mathbf{x}} \leq \vec{\mathbf{b}}$ *is integral.*

Proof: *Exercise.* □

We note that being TDI or not is not a characteristic of the polyhedron under consideration, but of the linear system used to describe that polyhedron. For instance, the polyhedron in Figure (2) can be described by System (6) and System (7). However, only System (6) is TDI.

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} . \leq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} . \leq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7}$$

Proof: *We first show that System (7) is not TDI. Let* $\vec{\mathbf{c}} = [c_1, c_2]$ *be an integral vector. Taking* $\{\max c_1.x_1 + c_2.x_2 :$ *System (7)*$\}$ *as the primal, the dual is:*

$$\begin{aligned} \min \vec{\mathbf{y}}.\vec{\mathbf{b}} \\ y_1 + y_2 = c_1 \\ y_1 - y_2 = c_2 \\ y_1, y_2 \geq 0 \end{aligned} \tag{8}$$

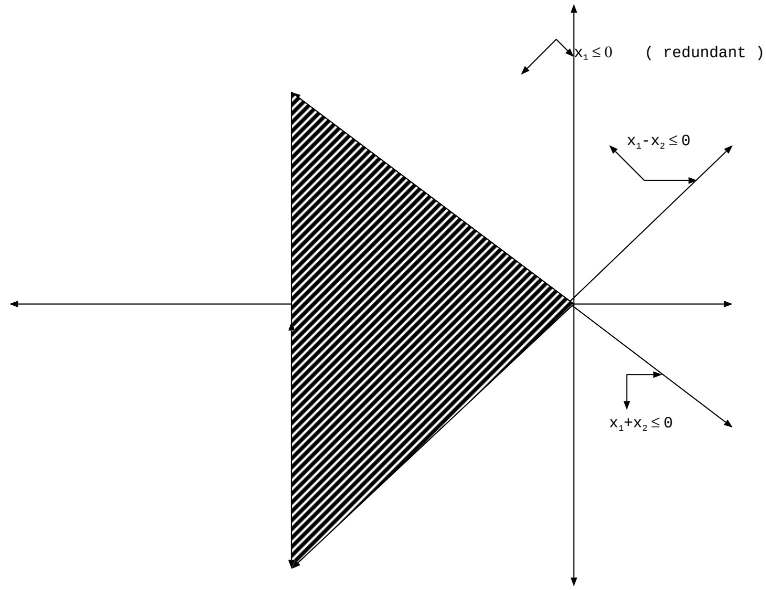

Figure 2: Polyhedron with redundant inequality

The only solution to this system is:
$y_1 = \frac{c_1+c_2}{2}$ and $y_2 = \frac{c_1-c_2}{2}$. If c_1 is odd and c_2 is even, then both y_1 and y_2 are fractional, thus proving that System (7) is not TDI. (Exercise: If $c_1 < c_2$ the dual is infeasible. Give an explanation based on Figure (2)!)

If the polyhedron is described by System (6), the dual is:

$$\begin{aligned} \min \vec{\mathbf{y}}.\vec{\mathbf{b}} \\ y_1 + y_2 + y_3 = c_1 \\ y_1 - y_3 = c_2 \\ y_1, y_2, y_3 \geq 0 \end{aligned} \tag{9}$$

The chief difference between System (9) and System (8) is that System (9) has an extra variable (y_2) that allows us some freedom. We consider the following cases:

1. *Both c_1 and c_2 even: Choose y_2 to be 0 and you have one single integral solution for y_1 and y_3;*
2. *Both c_1 and c_2 odd: Choose y_2 to be 0 and you have one single integral solution for y_1 and y_3;*
3. *c_1 odd and c_2 even: Choose $y_2 = 1$ and you have one single integral solution for y_1 and y_3, because $c_1 - y_2$ is even;*
4. *c_1 even and c_2 odd: Choose $y_2 = 1$ and you have one single integral solution for y_1 and y_3.*

In all cases, we can find an integral optimum for the dual and hence the primal system is totally dual integral. □

References

[KV00] B. Korte and J. Vygen. *Combinatorial Optimization.* Number 21. Springer-Verlag, New York, 2000.