CS 491I Approximation algorithms
Solutions of the Midterm - Practice Questions
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1. Suppose my linear program $A \cdot \vec{x} \leq \vec{b}$ has the specification that all variables except x_1 are ≥ 0; x_1 is an unrestricted variable. How would you model it in the standard form with all variables ≥ 0.

Answer:

We can represent the variable x_1 as a difference of two positive variables $x_1^{'}$ and $x_1^{''}$.

$x_1 = x_1^{'} - x_1^{''}$

$x_1^{'}, x_1^{''} \geq 0$

Depending which of the new variables is greater x_1 can be positive, negative or zero it they are equal. If the original system has *n* variables the new system will have *n* + 1 variables.

2. Suppose that all *n* variables are unrestricted variables. Show that you can replace this set with a set of *n* + 1 variables that are constrained to be non-negative.

Answer:

Each of the variables can be represented in the following form:

$x_i = x_i^{'} - d, x_i^{'} \geq 0, d \geq 0, i = 1,...,n.$

In order to show that this representation faithfully represents the original system lets assume that the optimum of the original LP is $\overrightarrow{x^*}$, where some of the components can be less than zero. We can set $d = \max(|x_i^*|)$, where | . | denotes absolute value. Then there exists an assignment of values to the variables $x_i^{'*}$ such that they will all be ≥ 0.

$x_i^{'*} = d + x_i^*$.

3. Show that a polytope defined in the usual way is closed.

Answer:

Lets assume that the polytope $P = \{\vec{x} : A\vec{x} \leq \vec{b}\}$ is open. Then there must exist a point, which is a closure of *P*, and does not belong to *P*. So, let $\overrightarrow{x^{'}} \in clP \setminus P$. The fact that $\overrightarrow{x^{'}}$ does not belong to *P* implies that at least one of the inequalities of the system does not hold for $\overrightarrow{x^{'}}$. Without any loss of generality we can assume that this is the first inequality. So, we have: $\overrightarrow{a_1} \cdot \overrightarrow{x^{'}} > b_1$. This can be rewritten as $\overrightarrow{a_1} \cdot \overrightarrow{x^{'}} - b_1 \geq \varepsilon > 0$. Hence there exist a ε-ball around $\overrightarrow{x^{'}}$ which does not contain points from *P*, that is: $\overrightarrow{x^{'}}$ is not a closure of *P*, and consequently, *P* must be closed.

4. Show that the set of optimal points of a Linear Program is a convex set.

Answer:
Let $\vec{x_1}, \vec{x_2}$ be two optimal points of a LP $\max\{\vec{c}\vec{x} : A\vec{x} \leq \vec{b}\}$. We know that $\vec{c}\vec{x_1} = \vec{c}\vec{x_2} = m$, where m is the optimal value of the LP. Then we should prove that any convex combination of these points is also optimal.

$\vec{z} = \lambda\vec{x_1} + (1-\lambda)\vec{x_2}, \lambda \in [0,1]$

$\vec{c}\vec{z} = \lambda\vec{c}\vec{x_1} + (1-\lambda)\vec{c}\vec{x_2} = \lambda m + (1-\lambda)m = m$

Which proves the claim.

5. Can a pivot of the Simplex Algorithm move the feasible point in the basis, while leaving the cost unchanged.

Answer:

It is possible for a LP to have two or more points that are optimal, so moving from one to another optimal point would not change the cost. But this does not happen because in every step we are choosing a pivot that is going to improve the result (for which the derivative of the cost function is > 0).

6. Prove or disprove: If a LP is unbounded then there exists a vector $\vec{\alpha}$ such that for any feasible $\vec{x}, \vec{x} + k\vec{\alpha}$ is also feasible, for all $k > 0$.

Answer:

We will prove by contradiction. Assume that LP is unbounded, but there does not exist a vector $\vec{\alpha}$ such that for any feasible $\vec{x}, \vec{x} + k\vec{\alpha}$ is also feasible, for all $k > 0$. This means that for all feasible points there exists a bound in every possible direction, that is for any vector $\vec{\alpha}$ there is a $k > 0$ for which $\vec{x} + k\vec{\alpha}$ is infeasible. This contradicts the assumption that the LP is unbounded.

7. Use the Simplex Method to solve:

$$\max z = 3x_1 + 10x_2 + 5x_3 + 11x_4 + 7x_5 + 14x_6$$
$$x_1 + 7x_2 + 3x_3 + 4x_4 + 2x_5 + 5x_6 = 42$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Answer:

The dual will be simpler to solve.

$$\min w = 42y_1$$
$$y_1 \geq 3$$
$$7y_1 \geq 10$$
$$3y_1 \geq 5$$
$$4y_1 \geq 11$$
$$2y_1 \geq 7$$
$$5y_1 \geq 14$$

The optimum is obtained for $y_1 = \frac{10}{7}, \min w = 42 \cdot \frac{10}{7} = 60$. In order to find the optimal $\vec{x}$ we use the Complementary Slackness Theorem. We se that the second inequality is binding, this implies that the optimal vector has $x_1 = x_3 = x_4 = x_5 = x_6 = 0$. Thus we get the following system.

$$\max z = 10x_2$$
$$7x_2 = 42$$

$\Rightarrow x_2 = 6, \max z = 60 = \min w, \overrightarrow{x^*} = [0,6,0,0,0,0]$.

8. Show that the point $\vec{x} = [10,0,16,6]$ is an optimal solution to the problem:

$$\max z = x_1 + 2x_2 + 5x_3 + x_4$$
$$x_1 + 2x_2 + x_3 - x_4 \leq 20$$
$$-x_1 + x_2 + x_3 + x_4 \leq 12$$
$$2x_1 + x_2 + x_3 - x_4 \leq 30$$
$$x_i \geq 0, i = 1,2,3,4$$

Answer:

The optimal solution of a LP is in the form $\vec{x} = \begin{pmatrix} \overrightarrow{x_B} \\ 0 \end{pmatrix} = \begin{pmatrix} B^{-1}\vec{b} \\ 0 \end{pmatrix}$. We can see that x_1, x_2, x_4 are in the basis. So we have:

$$B = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix}, \; B^{-1} = \frac{\begin{bmatrix} -2 & 1 & -3 \\ 0 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix}^T}{2} = \begin{bmatrix} -1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

Check: $\begin{bmatrix} -1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{1}{2} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ OK.

$$B^{-1}\vec{b} = \begin{bmatrix} -1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 20 \\ 12 \\ 30 \end{bmatrix} = \begin{bmatrix} 10 \\ 16 \\ 6 \end{bmatrix} = \overrightarrow{x_B}$$

$\Rightarrow \vec{x} = [10,0,16,6]$ is an optimal solution to the problem.

9. Consider the following linear program:

$$\begin{aligned} &\min z = \vec{b}\vec{w} - \vec{c}\vec{x} \\ &A\vec{x} \le \vec{b} \\ &A^T \vec{w} \ge \vec{c}^T \\ &\vec{x} \ge 0 \\ &\vec{w} \ge 0 \end{aligned}$$

where A is $m \times n$, $\vec{b}$ is $m \times 1$, $\vec{c}$ is $n \times 1$. Show that the optimal objective value is 0 or the problem is infeasible.

Answer:

We notice that we can split the system in two independent parts:

$$\begin{aligned} &\max \vec{c}\vec{x} \\ &A\vec{x} \le \vec{b} \\ &\vec{x} \ge 0 \end{aligned} \quad , \quad \text{and} \quad \begin{aligned} &\min \vec{b}\vec{w} \\ &A^T \vec{w} \ge \vec{c}^T \\ &\vec{w} \ge 0 \end{aligned}$$

And that they represent a primal LP and its dual. Because of the Strong Duality Theorem and the Corollary of possible choices: if they have optimal solutions they will be equal and the original system will have optimal objective value 0. Otherwise the original system will be infeasible.