CS 4911 Approximation algorithms
Solutions of the Quiz Problems
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1. Show that the set \( S = \{ A \cdot \vec{x} \{ \leq \} \vec{b} , \vec{x} \geq \vec{0} \} \) is convex.

Answer:
We will show the convexity of the \( \{ \leq \} \) form of the set. Any other form can be converted to this form. Let \( \vec{x}_1, \vec{x}_2 \in S \) and \( \lambda \in [0,1] \). We should prove that any linear combination of these vectors also belongs to the set \( S \).

\[
\vec{z} = \lambda \vec{x}_1 + (1 - \lambda) \vec{x}_2 \geq \vec{0} \Rightarrow \vec{z} \geq \vec{0}
\]

\[
A \cdot \vec{z} = \lambda A \cdot \vec{x}_1 + (1 - \lambda) A \vec{x}_2 \leq \lambda \vec{b} + (1 - \lambda) \vec{b} = \vec{b} \Rightarrow A \cdot \vec{z} \leq \vec{b}
\]

\( \Rightarrow \vec{z} \in S \), and thus \( S \) is convex.

2. Solve graphically:

\[
\begin{align*}
\text{min } z &= 4x_1 + 5x_2 \\
3x_1 + 2x_2 &\leq 24 \\
x_1 &\geq 5 \\
3x_1 - x_2 &\leq 6 \\
x_1, x_2 &\geq 0
\end{align*}
\]

Answer:
The system is infeasible.
3. Show that the halfspace \( H^- = \{ x : \vec{a} \cdot \vec{x} \leq \alpha \} \) is convex.

**Answer:**

Let \( \vec{x}_1, \vec{x}_2 \in H^- \) and \( \lambda \in [0,1] \).

\[
\vec{z} = \lambda \vec{x}_1 + (1 - \lambda) \vec{x}_2 \\
\vec{a} \cdot \vec{z} = \lambda \vec{a} \cdot \vec{x}_1 + (1 - \lambda) \vec{a} \cdot \vec{x}_2 \\
\leq \lambda \alpha + (1 - \lambda) \alpha = \alpha \\
\Rightarrow \vec{z} \in H^-, \text{ and thus } H^- \text{ is convex.}
\]

4. Given three vectors, \( \vec{a} = [4,2]^T, \vec{b} = [-2,6]^T, \vec{c} = [2,5]^T \), illustrate graphically

(a) The set of all linear combinations of \( \vec{a}, \vec{b}, \vec{c} \),
(b) The set of all non-negative linear combinations of \( \vec{a}, \vec{b}, \vec{c} \),
(c) The set of all convex combinations of \( \vec{a}, \vec{b}, \vec{c} \).

**Answer:**

(a) The set of all linear combinations covers the whole plane.
(b) Represents a part of the plane between the vectors.
(c) A triangle between the vectors.